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Developing Temperature-Time Curves for Objects That Can Be Approximated by a Sphere, Infinite Plate, or Infinite Cylinder

The purpose of this paper is to outline in detail an analytical procedure for developing the temperature-time cooling curve for physical objects that are part of a refrigeration design problem, so that refrigeration design engineers may predict cooling rates more accurately.

The selection of a method of analysis depends on the initial data and the use of the analytical results. Several of the available analytical methods of analyzing transient state heating or cooling data were contrasted by Pflug and Blaisdell.[1] There is also a number of response charts plotted from theoretical relationships available for analysis of transient state heat transfer problems (Williamson and Adams;[2] Gurney and Lurie;[3] Gröber;[4] Schack;[5] Heisler;[6] Schneider;[7] among others). In general, all methods have advantages; however, we believe that our method, based on the approach of Ball[8] and Ball and Olson,[9] offers the refrigeration engineer the most advantages with minimum disadvantages. This method has these advantages: (1) it can be used to correlate experimental data, (2) it can be used to predict heating and cooling times, and (3) it can be used to develop heat transfer and physical properties data.

This presentation is in several parts: description of the use of the method, outline procedure for prediction, relationship with fundamental heat transfer concepts, developing product data, examples of the solution of cooling problems, and discussion of the use and special features of the method.

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DESCRIPTION AND USE OF THE METHOD

This method is based on the concept that when the logarithm of the difference in temperature of an object and the cooling media are plotted vs. time, a curve of the general shape shown in Fig. 1 is generated. When the time function is large, the curve becomes coincident with the straight line asymptote which can be described by two parameters (Ball[8]); a direction function f, the time for the temperature difference between object and cooling media to decrease by 90% graphically on semi-logarithmic paper the time for the curve to traverse one log cycle, and a j or intercept function. The initial temperature T_o and cooling media temperature T_1 are design refrigerator and initial product conditions. The equation of the straight line asymptote is:

$$\log (T - T_1) = (-t)/(f) + \log j\,(T_o - T_1) \qquad (1)$$

The terms f and j are the parameters in Eq. 1 that relate the geometry and dimensions of the object, the thermal characteristics of the object, and the heat transfer conditions between refrigerant and object. In this paper three basic geometries will be treated: the sphere, the infinite plate or slab, and the infinite cylinder. (It is possible to combine these data to make other solutions.)

The curves in Figs. 2, 3, 4 and 5 relate f and j with product cooling conditions. Fig. 2 relates the ratio $f\alpha/a^2$ (r^2 instead of a^2 for sphere and cylinder) for an infinite slab, sphere, and infinite cylinder with the Biot number, (ha)/(k). Figs. 3, 4 and 5 relate the lag fac-

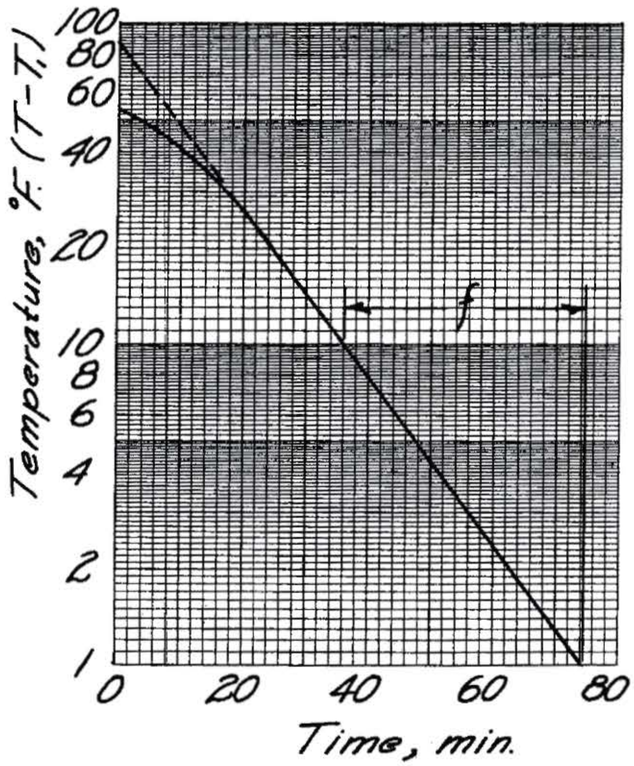


Fig. 1 Cooling curve in convenient form for theoretical use, with temperature scale expressed in degrees difference between processing temperature T_1 and temperature of product

tor j and the Biot number for the infinite cylinder, sphere, and infinite slab; Fig. 3 for the center of the object (j_c); Fig. 4 for the point representing the mean temperature of the object (j_m); and Fig. 5 for the surface temperature (j_s).

Plotting Cooling Data

The initial ingredient is a quantity of 2 or 3-cycle semi-logarithmic paper [if $(T_o - T_1) < 50F$ use 2-cycle, if $(T_o - T_1) > 50F$ use 3-cycle semi-logarithmic paper]. Tracing paper has advantages for direct comparison of graphs; however, drawing paper also can be used. One of the most important concepts of this method is the plotting of the logarithm of the difference in temperature of the object and the cooling media vs. time [$\log (T - T_1)$ vs. t]. On semi-logarithmic paper this is accomplished, as in Fig. 1, by plotting $(T - T_1)$ on the log scale vs. time. It is perhaps obvious at this point that there is an alternative to calculating the value of $(T - T_1)$ at each time value in order to plot $(T - T_1)$ on the log scale; the alternative is to design the log scale of the graph so that T_1 is added to make the scale $\log (T - T_1) + T_1$. The value of T can now be plotted directly on this new scale, as in Fig. 6. The equation of the cooling curve in Fig. 6 is:

$$\log (T - T_1) + T_1 = (-t)/(f) + \log j\,(T_o - T_1) + T_1 \quad (2)$$

We believe that engineers and scientists working with practical problems will have considerably more familiarity with the data, and those not regularly working in the area will be able to understand with less study when the data are plotted as in Fig. 6.

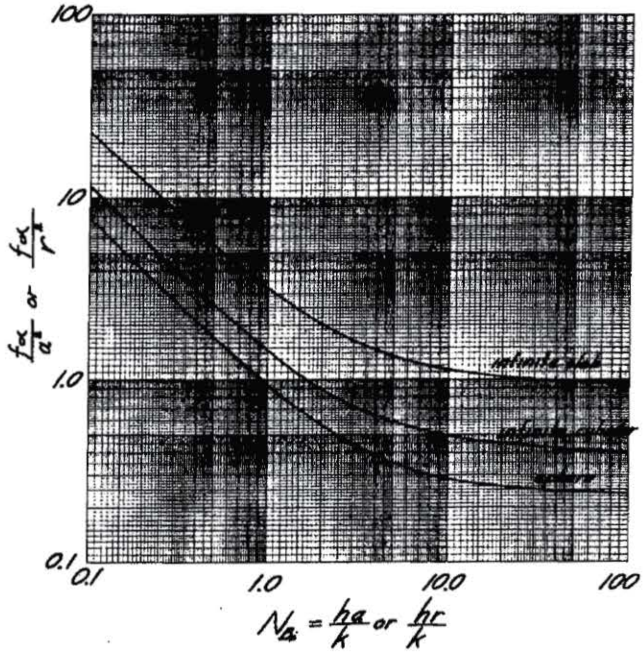


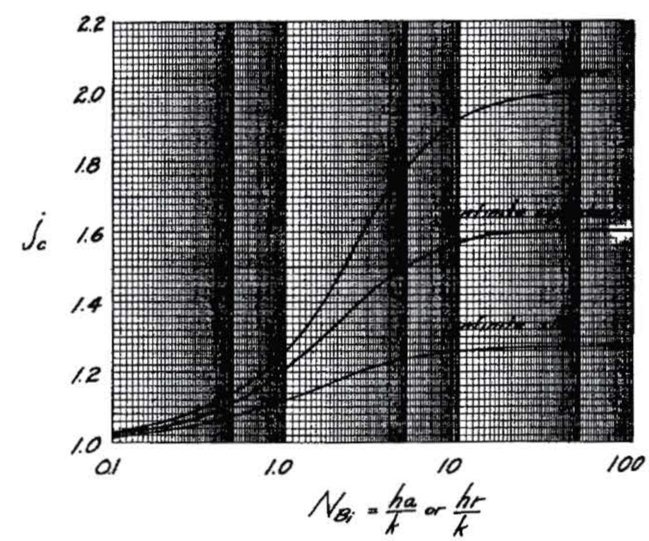
Fig. 2 Relationship of modulus $f\alpha/r^2$ or $f\alpha/a^2$ and N_{Bi} for infinite slab, cylinder, and sphere

The mechanics of setting up the temperature scale, as in Fig. 1, is as follows: Using 2-cycle paper, the bottom line will have a value of 1.0; the line at the top of the first cycle a value of 10; and the line at the top of the second cycle, 100. If the cooling medium temperature T_1 is 32F, the bottom line for the curve in Fig. 6 has a value of 33F, the second line 42F and the top line 132F. For design purposes, the intercept value T_a of the asymptotic cooling curve in Fig. 6 can now be calculated using the equation

$$j = (T_a - T_1)/(T_o - T_1) \quad (3)$$

and, using the intercept T_a, and the direction of the curve f, the cooling curve is drawn on the graph. For analysis purposes, the cooling temperature-time data are plotted directly using the new scale. A straight line is drawn through the data points and the f determined and the intercept ratio j calculated.

Fig. 3 Lag factor j at the geometric center as a function of N_{Bi}



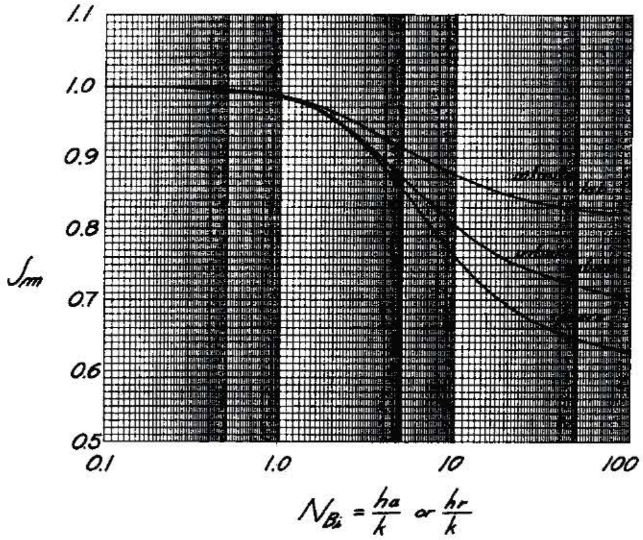


Fig. 4 Lag factor for the mean temperature in the body as a function of N_{Bi}

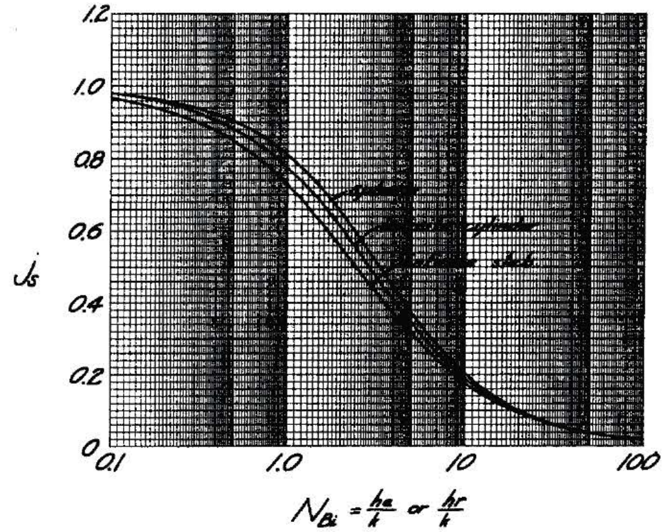


Fig. 5 Lag factor for the surface temperature of the body as a function of N_{Bi}

The semi-logarithmic cooling curve is unusual in several ways; the slope of the asymptotic cooling curve is the same regardless of the point of measurement in a conduction cooling object of the initial temperature difference ($T_o - T_1$), and the system can be used equally well for heating and cooling. In plotting heating data, the semi-logarithmic graph paper is usually turned upside down so that the curve rises as the temperature rises. Since the T_1 is greater than T, the equation is rearranged to

$$T_1 - \log(T_1 - T) = (t)/(f) + T_1 - \log j\,(T_1 - T_o) \quad (4)$$

The top line then will be 1, the line at the bottom of the first cycle 10, and at the bottom of the second cycle 100; if the heating medium T_1 is 180F, the corrected scale top line will be 179, the second 170, and the third line 80F.

OUTLINE PROCEDURE FOR PREDICTION

The parameters f and j are the links between product and refrigerator conditions and the cooling curve. Below, listed in outline form, are the steps that must be followed and intermediate data necessary to arrive at values for these parameters.

1. Determine the object size and select the appropriate geometry.
2. Calculate
$$N_{Bi};\ N_{Bi} = (ha)/(k) \quad (5)$$
3. Calculate the thermal diffusivity α;
$$\alpha = k/\rho C_p \quad (6)$$
4. Determine f from Fig. 2.
5. Determine j_c, j_m and j_s from Figs. 3, 4 or 5 as appropriate.
6. Using 2 or 3-cycle semi-logarithmic paper, label the temperature on the logarithmic scale as $[(T - T_1) + T_1]$ for cooling, $[T_1 - (T_1 - T)]$ for heating, and time on arithmetic scale.
7. Using refrigerator design conditions, cooling medium temperature T_1, and object initial temperature T_o, calculate the intercept temperature T_a for both center T_{ac}, mean T_{am}, and surface T_{as}, using Eq. 3.
8. Using the intercepts (j_c, j_m, j_s) and direction (f), the cooling curves for the center, mean, and surface can now be constructed.
9. Heat removed or remaining can be calculated from temperatures read directly from the cooling curve plotted using f and j_{mean}.
$$q\ (\text{heat removed}) = (T_o - T)\,C_p\rho V \quad (7)$$
$$q\ (\text{heat remaining}) = (T - T_1)\,C_p\rho V \quad (8)$$

RELATIONSHIP OF THE METHOD WITH FUNDAMENTAL HEAT TRANSFER CONCEPTS

The procedure developed in this paper for treating transient conduction heat transfer problems is a first term approximation producing a true asymptotic solution, and is based on the general approach of Ball and Olson.[9] The exact solutions are listed in the appendix of this paper, along with their first term approximations (Eqs. 3a, 11a and 19a), the simplifying equations, and the numerical solutions to these equations as computed, which were used to develop Figs. 2, 3, 4 and 5. These solutions are exact for spherical, cylindrical, or rectangular bodies. The initial temperatures and thermal conductivities are assumed to be uniform and the environmental temperature to change suddenly to a new constant level. The thermal properties and film coefficient are presumed uniform and constant during heating or cooling.

The curves obtained from Eqs. 3a, 11a and 19a are straight lines when $\log(T - T_1)$ or a related form is plotted vs. time. It is this fundamental relationship of semi-log temperature plot and a straight line cooling curve that provides the basis for plotting transient conduction heat transfer data in some form of $\log(T - T_1)$ vs. time, and drawing a straight line through the points. These solutions can be divided into three parts: a term giving the intercept of the straight line asymptote at the center, a position term that adjusts the intercept for cooling conditions measured at a distance x from the coordinate axis, and the exponential part of the equation which contains the time function. In a plot of $\log(T - T_1)$ vs. time, the exponential function, $-\beta_1^2\alpha/2.303a^2$ is the true slope

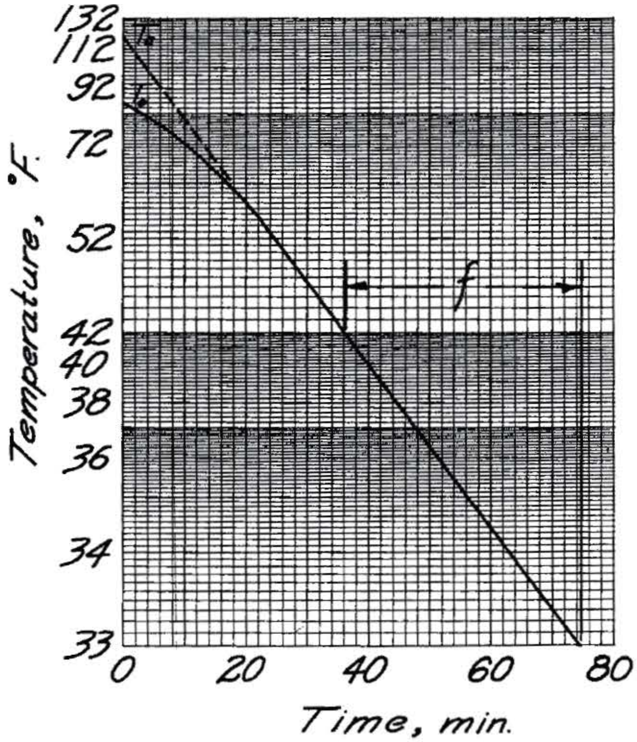


Fig. 6 Cooling curve in convenient form for practical application

($\tan \phi$) of the straight line asymptote. In general, the straight line asymptote is within 5% of the true cooling curve at values of time greater than 0.5 to 0.8 f for N_{Bi} of about 1.0 to 100 or $(T - T_1)/(T_o - T_1) < 0.3$.

Composite Shape

Solutions for other regular shapes may be obtained from those of the infinite slab and/or the infinite cylinder for prescribed surface temperature or film coefficient if the thermal conductivity is isotropic. Composite solutions are the product of the respective individual direction solution given by Eqs. 16, 23, 30, hence for a solid with heat transfer in three directions

$$1/f_{composite} = 1/f_1 + 1/f_2 + 1/f_3 \quad (9)$$

$$j_{composite} = j_1 \cdot j_2 \cdot j_3 \quad (10)$$

Rutov[10] has suggested a means of improving the estimates for slightly irregular composite shapes. For Biot numbers of 1 and below, the film coefficient h in the Biot number is multiplied by the ratio of the actual surface area to that of the regular body of equivalent volume. For high Biot numbers, the thermal conductivity in the Biot number and in the apparent thermal diffusivity is multiplied by this ratio. This approximation for prediction or analysis of cooling curves will usually require measurement of the actual products, and approximation of their area by frustums of cylinders or cones or sectors of spheres because product shapes and areas have infrequently been reported.

DEVELOPING DATA FOR USE WITH THE METHOD

Convection Surface Transfer Coefficient h

The surface transfer coefficient affects both the cooling rate parameter f, as well as the cooling lag j. Surface heat transfer coefficients on flat or cylindrical surfaces exposed to flow may be calculated by equations from the ASHRAE Guide And Data Book or standard texts, such as McAdams.[11] But material of known physical properties can be used to evaluate h if velocities are not known and cannot be easily measured, or if correlations are not available for the downstream side of the objects. When h has been evaluated it can be used to develop curves for cooling design, for comparing product cooling data, and for analyzing cooling data for product physical properties.

Example of Use of Transient Cooling Procedure to Determine h

An accurate value of the h of a small diameter cylindrical meat product cooled in an air blast is needed. A 1-in. dia by 8-in. long copper cylinder with insulated ends is available as a transducer. Physical properties of the copper cylinder are: $k = 226$ Btu/hr ft F, $C_p = 0.0915$ Btu/lb F, $\rho = 0.327$ lb/in.^{3}; the f of the cylinder in the air blast room is 15.2 min.

Solution for h: The value of $f\alpha/r^2 = 637$, from the $f\alpha/r^2$ vs. N_{Bi} plot for infinite cylinder (we will consider our isolated 1 by 8 in. an infinite cylinder) in Fig. 2 we find that $N_{Bi} = 0.0018$ and $h = 9.75$ Btu/hr ft^2 F.

PRODUCT PHYSICAL PROPERTIES

Density

The densities ρ of food products are generally not available in handbooks for use by the food or refrigeration engineer. The density of any object can be evaluated using the Archimedes principle after weighing in two fluids, or by displacement. For some products, such as tray-packed apples, ρ can be estimated from the bulk density of the net package, the count of the fruit, and the average dimensions of the fruit.

The formula in the ASHRAE Guide And Data Book for prediction of the density of ice cream mix suggests that the specific gravity can be predicted for other products using the widely published values for the moisture, fat, protein, carbohydrate, crude fiber, and ash content of foodstuffs. Using approximate values for specific volume increase or density for soluble and insoluble components we obtain:

$$\text{Specific gravity} = \frac{100}{\text{sum of}\left[\dfrac{\%\text{ of component}}{\text{its density}}\right]} \quad (13)$$

where specific gravity of
moisture = 1.00
fat = 0.93
protein = 1.37
carbohydrate = { 1.48 for starchy products; 1.61 for sugars }
fiber = 1.35
ash = 1.23

This equation appears to give reasonable density values for meat, fatty substances, and even potatoes; however, it cannot be used for products such as apples and woody vegetables, which contain air between cells, because it predicts a specific gravity of about 1.1 when in fact the actual product is less dense than water. Since simple, direct methods for measuring density are available, it is suggested that the density of the product be measured if accurate values are not available.

Specific Heat

The specific heat (C_p) of a food product can be simply determined in calorimeters (Short[15]). Experimental values for food products have also been determined by Ordinanz,[30] Staph and Woolrich,[12] Mannheim, et al.[13] and Moline, et al.[14] Most of the published values of specific heat, as tabulated by Andersen[22] and the ASHRAE Guide And Data Book,[16] are based on Siebel's formulas[17] for non-fatty substances:

(above freezing) $C_p = 0.008 \times (\text{per cent moisture}) + 0.20$ (11)

(below freezing) $C_p = 0.003 \times (\text{per cent moisture}) + 0.20$ (12)

Standard methods for determining the moisture content of these and other food products are listed in A.O.A.C.[18]

Thermal Conductivity

Thermal conductivity k is a more elusive and variable physical property than either density or specific heat; consequently, thermal conductivity data are scarce. The available thermal conductivity data have been assembled on a product basis: data for meat products are summarized by Lentz[19] and Miller and Sunderland;[30] for poultry products by Wallers and May (1963); for fats by Lentz;[19] and for fruits and vegetables by Bennett.[21]

The calculation of k from generally available product properties has been the object of studies by several researchers. Andersen[22] suggested that k be estimated using the equation

$$k_{overall} = [k_{water} \times (\text{per cent water}) + (100 - \text{per cent water}) \times k_{solid}]/100 \quad (14)$$

Thermal conductivities calculated using this formula do not agree with the tabulated food product k values of Smith, et al.[23] or Bennett.[21] The k values and thermal diffusivity for water at several temperatures are listed in Table I. It is our opinion that Andersen's method can be used to calculate thermal conductivities of true solutions, as typified by solutions of sucrose in water (Riedel[24]). We do not believe that the Andersen method should be used to calculate thermal conductivities of objects or systems made up of particles of one material dispersed in a second material or in cellular material which contains air in the intercellular spaces. In these complicated systems, the method of Lentz[19] appears to be the best available. He used the equation of Euchen (1940) for spherical particles dispersed in a continuous phase for ice spheres in frozen meat

$$k = k_c \left[\frac{1 - [1 - C_k(k_d/k_c)]}{1 + (C_k - 1) C_d} C_d \right] \quad (15)$$

This method is accurate for small concentrations (C_d values) of the dispersed phase only. It appears that air may be considered a dispersed phase in fruits and vegetable products. The structure of products like woody vegetables may cause the fiber and cell contents to be nearly parallel heat conductors in the direction of the grain, but more nearly series conductors perpendicular to the grain. The problems of differences in structure, in addition to differences due to normal biological variation, complicate the problem of evaluating k in biological systems.

Thermal conductivity can be measured in the laboratory by the guarded hot plate method (Lentz[19]) by the modified Cenco-Fitch method (Bennett, et al.[25] and Walters and May[26]) or from transient cooling.

Measurement of the Effective Thermal Conductivity

The effective thermal conductivity of the whole product may be determined in the field as part of the analysis of transient cooling or heating data and, as such, offers an advantage to the heat transfer evaluation. In the transient method, the f and j of the cooling or heating curve of the specific geometry of product is used with the appropriate $f\alpha/r^2$ vs. N_{Bi} curve to evaluate the N_{Bi} and the thermal diffusivity ($\alpha = k/\rho C_p$) and then, assuming ρ and C_p have been evaluated, k can be calculated. It is recommended that the analysis for k be based on reliable experimental values of h and f, since h by calculation is more reliable than j (which incidentally is affected by initial temperature distribution, among other things, Ball and Olson[9]).

When the objective of the experiment is to determine physical properties, it is preferable to determine f under conditions where β_1 is insensitive to changes in h and, therefore, in the Biot number. When β_1 approaches π (for a sphere), the N_{Bi} increases so fast (see Eq. 10a) that large changes in h have a negligible effect on β and, therefore, on the thermal diffusivity (see Eq. 12a). This can be illustrated by considering three spherical shaped products of 0.1-ft radius of widely different thermal diffusivities. Let the materials have properties near those of copper, water, and insulation material with thermal conductivities of 229

Table I. Thermal Conductivities and Diffusivities of Water, after Kreith[31]

Temperature T, F	Thermal Conductivity k, Btu/hr ft F	Thermal Diffusivity α, in.^{2}/min $\times 10^2$
32	0.319	1.217
40	0.325	1.250
60	0.340	1.313
80	0.353	1.363
100	0.364	1.411
150	0.384	1.505
200	0.394	1.572
250	0.396	1.606

Table II. Comparison of the h Values Necessary to Produce Root Equation β Values Near π for Three Materials of Widely Differing k Values

			h, Btu/hr ft^2 F		
β	N_{Bi}	h/k	Copper	Water	Expanded Polystyrene
3.10	75.5	755	172500	249	30
3.11	99.4	990	228000	328	40
3.12	145.0	1450	~	480	58
3.13	271.0	2710	~	884	108
3.14	1970.0	19700	~	6520	786

Btu/hr ft F, 0.33 Btu/hr ft F, and 0.0392 Btu/hr ft F, respectively. The data in Table II indicate that it is practically impossible to obtain N_{Bi} values corresponding to β values as large as 3.0 for materials, such as copper, which have large k values. For products having thermal conductivities similar to water, as do most foodstuffs, the h value resulting from heat transfer to or from water at moderate velocities will produce an N_{Bi} and β value greater than 3.0; for example, if h is between 300 and 500 Btu/hr ft F, the β root value is between 3.11 and 3.12. If we arbitrarily pick the value of 3.115, the error introduced in the diffusivity calculation (Eq. 12a) is at most $(3.115^2\text{-}3.11^2)/3.11^2 \times 100 =$ 0.33%. The f and j values for materials with low k values are less sensitive to changes in h. If condensing steam is employed in heating food products, the exterior film coefficient obtained will be sufficiently large; thus, for any material that has a thermal conductivity equal to or less than water, the β value can be assumed to be π. For such conditions, the accuracy of ρ and C_p are equally important in the determination of k. Due to the nature of highly conductive materials such as copper, the N_{Bi} encountered in practice will generally be low and the internal temperature will consequently be nearly uniform. Since the internal temperature gradient is negligible, the heating or cooling rate parameter f will be directly proportional to the C_p, and ρ and inversely proportional to the film coefficient h.

Example of the Determination of k of an Apple

Consider a Jonathan apple weighing 0.188 lb, a measured density of 51.2 lb/ft³, a measured moisture content of 83% and a specific heat (by the Siebel equation) of 0.86 Btu/lb F. The fruit is cooled in a water stream with a velocity of 90 fpm. The f of the cooling curve is 27.5 min, (0.4583 hr).

Solution: Using the correlation of Pigford and coworkers in Zenz and Othmer,[27] h is calculated to be about 500 Btu/hr ft F. The radius of the equivalent sphere is 0.0955 ft. As a first approximation, assume k of apple to be the same as water, 0.33 Btu/hr ft F, then

$$\text{the } N_{Bi} = \frac{500 \text{ Btu/hr ft}^2 \text{ F } (0.0955 \text{ ft})}{0.33 \text{ Btu/hr ft F}} = 144. \text{ From}$$

the sphere curve in Fig. 2, $f\alpha/r^2 = 0.236$, $\alpha =$

$$\frac{0.236(0.0955\text{ ft})^2}{0.4583 \text{ hr}} = 0.00467 \text{ ft}^2\text{/hr and k} = 0.00467$$

ft²/hr x 51.2 lb/ft³ x 0.86 Btu/lb F = 0.205 Btu/hr ft F.

Using the value k from the first approximation in the second iteration, $N_{Bi} = 232$, $f\alpha/r^2 = 0.2353$, $\alpha =$ 0.00464 ft²/hr, and k = 0.203 Btu/hr ft F.

EXAMPLES IN THE USE OF THE METHOD FOR SOLVING COOLING PROBLEMS

The three examples presented below were selected to show both the simplicity and the wide range of application of this method to cooling problems.

Example 1 Design Analysis

Apples are to be cooled from an initial temperature of 85 to 40F as they move through a tunnel on a conveyor belt. The apples can be cooled either by a cold water spray at 35F or by a 600 fpm air blast at 20F. Determine the relative cooling time for the two systems. (Apple dia 3.0 in., $C_p = 0.86$ Btu/lb F, $\rho = 51.2$ lb/ft³, k = 0.203 Btu/hr ft F.)

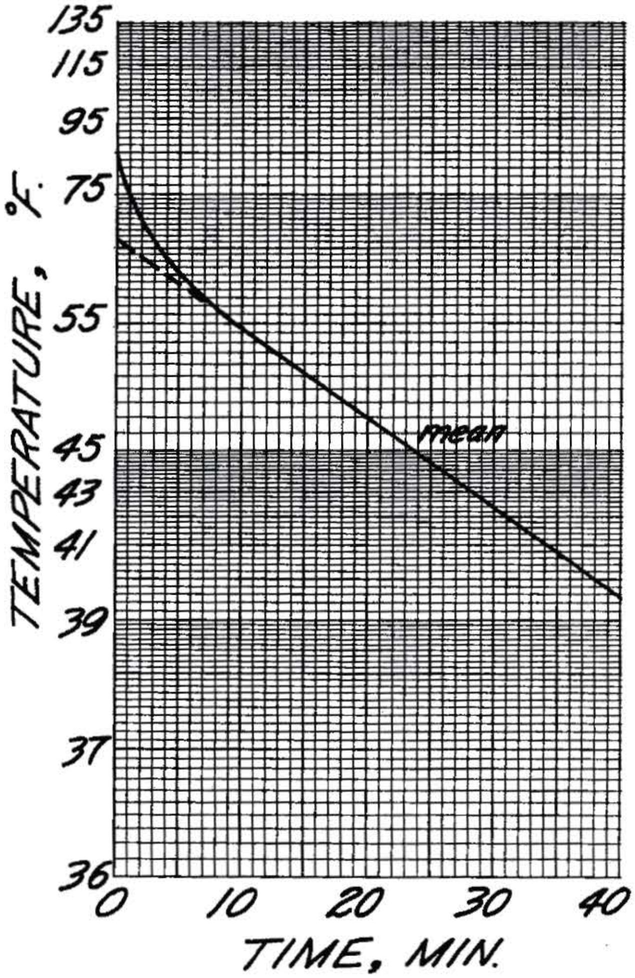


Fig. 7 Curve for cooling apples with water (Example 1)

Solution: Determine f and j_{mean} for water and air cooling, then evaluate cooling time, using Eq. 1 or plotted graphically as in Figs. 7 and 8.

Water: Flow rate 10 gpm/ft²; apple cross section 0.049 ft² then from Jakob (page 364 Fig. 37-5[28]) h ≅ 700 Btu/hr ft² F and $N_{Bi} = 430$; from Fig. 2, $f\alpha/r^2 =$ 0.2344, f = 47.5 min, and from Fig. 4, $j_m = 0.635$.

$$t = f \log \frac{j_m(T_1 - T_o)}{(T_1 - T)} = 38.1 \text{ min}$$

Air: Flow rate 600 fpm at 20F, $T_f = 40$F, $k_f =$ 0.0141 Btu/hr ft F, $\rho_f = 0.0794$ lb/ft³, μ_a 0.0414 lb/hr ft, $N_{Re} = 17{,}200$ from McAdams (page 265[11]), $h_m D_a/k_f$ $= 0.37\ (D_a \cdot V \cdot \rho_f/\mu_f)^{0.6}$; $h_m = 7.3$ Btu/hr ft² F, N_{Bi} = 4.4, from Fig. 2, $f\alpha/r^2 = 0.3665$, f = 74.3 from Fig. 4 $j_m = 0.87$.

$$t = f \log \frac{j_m(T_1 - T_o)}{(T_1 - T)} = 32.4 \text{ min}$$

The advantage of the air in this case is due to the lower air temperature (T_1); if the water temperature can be reduced to 34F, time for water cooling will be reduced.

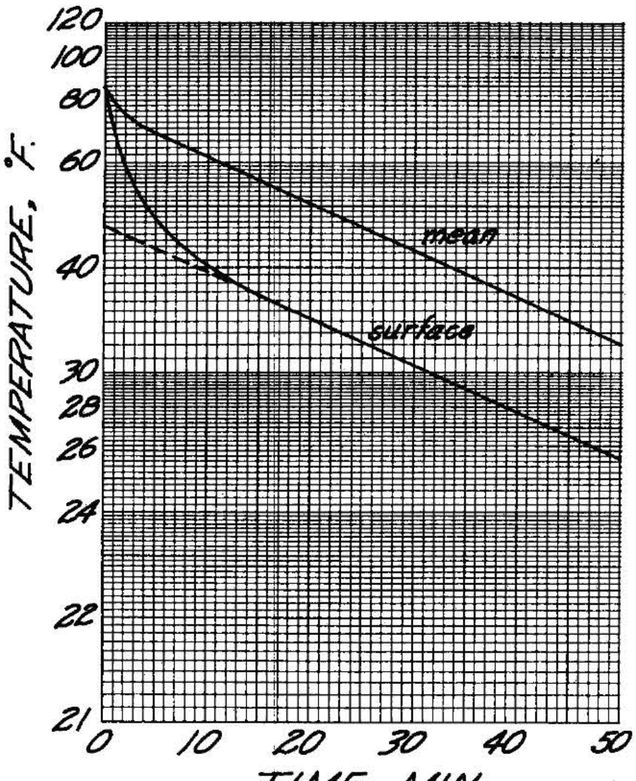


Fig. 8 Curve for cooling apples with air (Example 1)

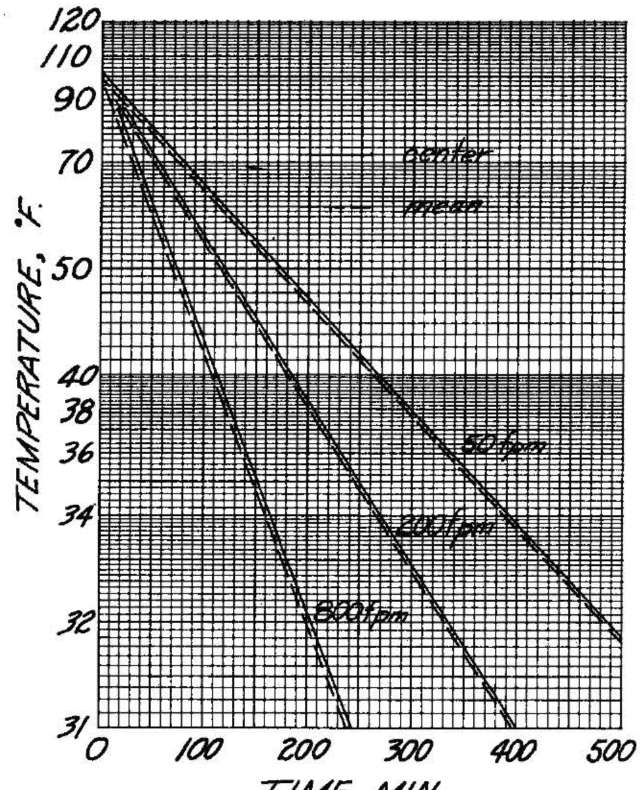


Fig. 9 Cooling curves for center and mean temperature for the three air velocities of Example 2

Example 2 Design Analysis

A food product in slab form 1 in. thick, 12 in. wide and 24 in. long is to be cooled. What is the effect of air velocity (50, 200 and 800 fpm) and how much heat will remain in the product if cooling is limited to a 2 hr cycle and a 4 hr cycle? The initial temperature of the product is 100F, cold air temperature 30F, thermal diffusivity of product 0.0127 in.^{2}/min, specific heat 1.0, weight per unit 10.5 lb, and the product thermal conductivity 0.33 Btu/hr ft F. The product initial temperature and properties are uniform throughout.

Solution: The critical part of the problem is determining h. Using the recommended formula of Table 5.2 of the ASHRAE Guide And Data Book and Table 26-4 of Jakob,[29] the h values for each were determined and the values for the 1-in. direction are shown in Table III. The N_{Bi} for a slab can now be calculated, $f\alpha/a^2$ determined from Fig. 3, and j_c and j_m from Figs. 3 and 4. The value f can now be calculated. With f, j_c and j_m, cooling curves for the three air velocities are developed in Fig. 9 and the data from these curves used to complete Table IV.

Table III. Example 2 Data Summary Sheet

Air velocity fpm	h Btu/hr ft^2 F	N_{Bi}	j center	j mean	f/a^2 min/in.^{2}	f min
50	1.2	0.151	1.023	0.999	1250	313
200	1.8	0.226	1.034	0.998	885	214
800	3.1	0.389	1.056	0.999	525	131

Example 3 Calculation of Cooling Curve Parameters for Complicated Geometry

A food product in a 4.0-in. OD by 4.5 in. long can is cooled in a constant temperature air blast (1000 fpm) at 35F. The physical properties of the food are: specific gravity = 1.08, specific heat 0.82 Btu/lb F, k = 0.282 Btu/hr ft F, thermal diffusivity = 0.0112 in.^{2}/min. The average h for the 4.0 by 4.5-in. can in the air blast is calculated to be 6.6 Btu/hr ft^2 F.

Solution: The composite j-value will be the product of the j for an infinite cylinder where r = 3.0/2, and the j for an infinite slab where a = 4.5/2. The composite f-value will be calculated as: $1/f_{composite} = 1/f_{r=4.0/2\ cylinder} + 1/f_{a=4.5/2\ infinite\ plate}$. The N_{Bi} for the cylinder is 3.9 and for the slab 4.4. The j_c values from Fig. 4 are 1.46 for the cylinder and 1.23 for the slab, giving a composite j_c of 1.80. The f calculated from Fig. 2 for the cylinder is 128 min and for the slab 625 min, giving a composite f of 106 min.

Table IV. Comparison of the Heat Removed in 2 and 4 hr

Cooling time	Air velocity fpm	Mean temperature F	Heat removed Btu $(T_o - T)\rho C_p V$*	Heat remaining Btu $(T - T_1)\rho C_p V$*
	50	57.8	443	292
2 hr	200	48.0	546	189
	800	38.3	648	87
	50	40.8	612	114
4 hr	200	34.8	674	50
	800	31.0	724	10

* $W = \rho V = 10.5$ lb
$C_p = 1$ Btu/lb F

DISCUSSION OF THE USE AND FEATURES OF THE METHOD

The method that we have described can be used to correlate experimental data, predict heating and cooling times, and develop heat transfer and physical property data for conduction heat transfer systems. The method as applied to real problems, rather than ideal situations, is most sensitive to limitations in the area of developing heat transfer and physical property data. As the product situation departs further from the ideal, the accuracy of predicting cooling rates becomes less precise and finally the ability of the method to correlate cooling data is affected; however, cooling data may generally be analyzed and correlated even when the assumptions implicit in the proposed method are not met. Cooling data for packaged products may be analyzed by this method, however it is questionable if cooling rates can be predicted for complicated product and package situations. The resulting f and j may be interpreted in terms of effective thermal conductivities and film coefficients, which include the influence of convection within the package as well as resistances of the product and package and their surfaces.

Changes in thermal or surface properties will cause the f value to approach that for conditions at the cooling temperature, but j will generally decrease because the f of the cooling curve will usually increase during cooling. The calculation of f at about three temperatures will make possible an improved estimate of the cooling curve.

Moisture loss from the product during cooling increases the effective N_{Biot} number of the product slightly, because the latent heat is absorbed from the product. The nature of the product, protective covering, and protective packaging will all affect rate of moisture loss.

Living foods, such as apples and pears, produce heat as a product of respiration. Awbery (1927) in Jakob[29] has shown that for cooling of single fruit, such as apples, this energy has a negligible effect on the cooling rate.

If the environmental temperature does not change quickly from the existing to the new temperature level and does not remain constant thereafter, the difference between product and environmental temperature can be used to develop the cooling curve; since the product temperature in the interior may lag far behind that of the surface, the environmental temperature used may be that measured at a time previous to the time the product temperature is measured. If the temperature change occurs over a short time compared to f, the temperature change will not affect f.

The form of the exact equations for more complex problems may often be reduced to the approximation of Eq. 1. These include irregularly shaped products like poultry, composite products like stone fruit, or products with a large seed cavity, and sometimes (Carslaw and Jaeger, p. 42, 1959) products which, like wood, have unequal thermal conductivities parallel and perpendicular to the grain.

Cooling of composite bodies may be calculated using Eqs. 9 and 10 as illustrated above. The values of f and j should be calculated for one, two, and three dimensional heat transfer, in order of importance, and sketched on a cooling curve to give a better estimate if the f values for each dimension differ appreciably. When the individual j values are less than one (as for j_m and j_s) for greatly differing f's, then estimation of the temperatures at these positions should not be attempted using this method for times short relative to the largest f value, because the second and third terms of the series solution must be considered.

The major source of error for the cooling problems described in this paper are the inadequacy of tabulated densities and thermal conductivities, and the difficulty of estimating the cooling medium with respect to the object being cooled and, in turn, the effective film coefficient of the product being cooled.

SUMMARY OF METHOD AND USE

The transient cooling procedures described are useful in developing product cooling graphs, in correlating cooling data, and for combining experimental cooling data and the analytical procedure to evaluate the surface transfer coefficient or product thermal conductivity in conduction heat transfer systems.

In using the method in design after the parameter f and j have been evaluated, any number of cooling medium temperature—product initial temperature combinations can be evaluated simply and quickly.

The method can be used to calculate the quantity of heat removed and remaining when the product geometry can be approximated by either an infinite slab, sphere, or infinite cylinder.

The method is, in general, analytical in character; however, the graphic cooling curve can be drawn making possible a better understanding of the cooling situation.

NOTATION

A	=	Surface area of product
2a	=	Thickness of infinite plate, width of brick, thickness of brick, length of finite cylinder
C_d	=	$V_d/(V_c + V_d)$; subscript C denotes the continuous phase, subscript d denotes the dispersed phase in Eq. 15
C_k	=	$3k_c(2k_c + k_d)$
C_p	=	Specific heat (at constant pressure) of product
D	=	Diameter of spherical or cylindrical product
e	=	Naperian base (= 2.71828 . . .)
f	=	Time required for the asymptote of the cooling curve to cross one log-cycle, the time required for a 90% reduction of temperature on the linear portion of the cooling curve (Ball[8])
h	=	Surface heat transfer coefficient
$J_n(\beta_n)$	=	Nth order Bessel function of first kind for the argument β_n
j	=	Lag factor $(T_a - T_1)/(T_o - T_1)$, dimensionless
k	=	Thermal conductivity
L	=	Characteristic length of product in the direction of fluid flow
N_{Bi}	=	Biot number ha/k; dimensionless (k is thermal conductivity of product)
N_{Nu}	=	Nusselt number h(L or D)/k; dimensionless (k is thermal conductivity of heat transfer medium)
N_{Re}	=	Reynolds number (L or D)$V\rho/\mu$; dimensionless
r	=	Radius of sphere or infinite cylinder
T	=	Temperature; T_o, initial temperature; T, product temperature; T_1, temperature of cooling or heating

media; T_a, the apparent initial temperature as defined by the linear portion of the cooling curve, that is, the ordinate value of the asymptote of cooling curve; T_{ac} for center, T_{am} for mean and T_{as} for surface; T_f, temperature of fluid film, $(T_s + T_1)/2$ where T_s is the surface temperature

- t = Time
- x = Distance from center of product to point of measurement
- V = Volume of product
- α = Thermal diffusivity $k/(C_p\rho)$
- β_n = Nth root of the boundary equation for a particular shape. The β_n for a slab, cylinder or sphere are different as shown in Tables IA, IIA, and IIIA
- ρ = density
- μ_f = fluid film viscosity

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APPENDIX

Infinite Plate

$$\frac{(T - T_1)}{(T_o - T_1)} = \sum_{i=1}^{\infty} \frac{2 \sin \beta_i}{\beta_i + \sin \beta_i \cos \beta_i} e^{\frac{-\beta_i^2 \alpha t}{a^2}} \cos\left(\beta_i \frac{x}{a}\right) \quad (1a)$$

$$\text{root equation: } N_{Bi} = \beta_i \tan \beta_i \quad (2a)$$

First term approximation:

$$\log \frac{(T - T_1)}{(T_o - T_1)} = \frac{-\beta_1^2 \alpha t}{2.303\, a^2} + \log\left[\left(\frac{2 \sin \beta_1}{\beta_1 + \sin \beta_1 \cos \beta_1}\right)\left(\cos\left[\beta_1 \frac{x}{a}\right]\right)\right] \quad (3a)$$

$$\frac{\beta_1^2 \alpha}{2.303\, a^2} = \frac{1}{f}; \quad \frac{f\alpha}{a^2} = \frac{2.303}{\beta_1^2}$$

or

$$\frac{f}{a^2} = \frac{2.303}{\beta_1^2 \alpha} \quad (4a)$$

$$j = \frac{2 \sin \beta_1}{\beta_1 + \sin \beta_1 \cos \beta_1} \cos\left(\beta_1 \frac{x}{a}\right) \quad (5a)$$

$$j_c = \frac{2 \sin \beta_1}{\beta_1 + \sin \beta_1 \cos \beta_1} \quad (6a)$$

$$j_m = \frac{2 \sin^2 \beta_1}{\beta_1(\beta_1 + \sin \beta_1 \cos \beta_1)} = j_c\left(\frac{\sin \beta_1}{\beta_1}\right) \quad (7a)$$

$$j_s = \frac{2 \sin \beta_1 \cos \beta_1}{\beta_1 + \sin \beta_1 \cos \beta_1} = j_c\ (\cos \beta_1) \quad (8a)$$

Sphere

$$\frac{(T - T_1)}{(T_o - T_1)} = \sum_{i=1}^{\infty} \frac{2(\sin \beta_i - \beta_i \cos \beta_i)}{\beta_i - \sin \beta_i \cos \beta_i} e^{\frac{-\beta_i^2 \alpha t}{r^2}} \frac{\sin\left(\beta_i \frac{x}{r}\right)}{\beta_i \frac{x}{r}} \quad (9a)$$

$$\text{root equation: } N_{Bi} = 1 - \beta_i \cot \beta_i \quad (10a)$$

Table IA. Numerical Solutions for Infinite Plate Eq. 3a

β_1	N_{Bi}	$\frac{f\alpha}{a^2}$	j_c	j_m	j_s
0.000	0.00000	∞	1.000000	1.000000	1.000000
0.020	0.00040	5756.46273	1.000067	1.000000	0.999867
0.030	0.00090	2558.42788	1.000150	1.000000	0.999700
0.040	0.00160	1439.11568	1.000267	1.000000	0.999467
0.050	0.00250	921.03404	1.000417	1.000000	0.999166
0.070	0.00491	469.91533	1.000817	0.999999	0.998366
0.090	0.00812	284.26976	1.001350	0.999999	0.997297
0.110	0.01215	190.29629	1.002016	0.999997	0.995960
0.130	0.01700	136.24764	1.002816	0.999994	0.994354
0.150	0.02267	102.33712	1.003749	0.999989	0.992478
0.220	0.04920	47.57407	1.008059	0.999947	0.983763
0.290	0.08654	27.37913	1.013993	0.999840	0.971653
0.360	0.13551	17.76686	1.021540	0.999618	0.956056
0.430	0.19721	12.45314	1.030683	0.999213	0.936856
0.500	0.27315	9.21034	1.041397	0.998545	0.913912
0.570	0.36535	7.08706	1.053643	0.997508	0.887063
0.600	0.41048	6.39607	1.059347	0.996921	0.874317
0.720	0.63149	4.44172	1.084756	0.993432	0.815526
0.840	0.93713	3.26330	1.113883	0.987435	0.743476
0.960	1.37122	2.49846	1.145865	0.977794	0.657176
1.080	2.02091	1.97410	1.179331	0.963074	0.555852
1.200	3.08658	1.59902	1.212226	0.941535	0.439258
1.320	5.15242	1.32150	1.241615	0.911190	0.308138
1.340	5.70253	1.28235	1.245911	0.905131	0.285006
1.380	7.14486	1.20909	1.253804	0.892067	0.237772
1.420	9.34519	1.14193	1.260617	0.877684	0.189377
1.460	13.12337	1.08021	1.266162	0.861916	0.139999
1.500	21.15213	1.02337	1.270241	0.844706	0.089853
1.540	49.99015	0.97090	1.272652	0.826005	0.039187
1.550	74.52165	0.95841	1.272969	0.821093	0.026471
1.554	92.51153	0.95348	1.273063	0.819101	0.021382
1.558	121.74704	0.94859	1.273136	0.817094	0.016281
1.562	177.56954	0.94374	1.273191	0.815071	0.011199
1.566	326.49735	0.93893	1.273225	0.813033	0.006107
1.570	1971.55200	0.93415	1.273239	0.810980	0.001014
$\pi/2$	∞	0.93320	1.273240	0.810569	0.000000

First term approximation:

$$\log\frac{(T-T_1)}{(T_o-T_1)} = \frac{-\beta_1^2\alpha t}{2.303r^2} + \log\left\{\left[\frac{2(\sin\beta_1-\beta_1\cos\beta_1)}{\beta_1-\sin\beta_1\cos\beta_1}\right]\left[\frac{\sin\left(\beta_1\frac{x}{r}\right)}{\beta_1\frac{x}{r}}\right]\right\} \quad (11a)$$

$$\frac{\beta_1^2\alpha}{2.303r^2} = \frac{1}{f};\quad \frac{f\alpha}{r^2} = \frac{2.303}{\beta_1^2} \quad (12a)$$

or

$$\frac{f}{r^2} = \frac{2.303}{\beta_1^2\alpha}$$

$$j = \left[\frac{2(\sin\beta_1-\beta_1\cos\beta_1)}{\beta_1-\sin\beta_1\cos\beta_1}\right]\left[\frac{\sin\left(\beta_1\frac{x}{r}\right)}{\beta_1\frac{x}{r}}\right] \quad (13a)$$

$$j_c = \frac{2(\sin\beta_1-\beta_1\cos\beta_1)}{\beta_1-\sin\beta_1\cos\beta_1} \quad (14a)$$

$$j_m = j_c\frac{3}{\beta_1^3}(\sin\beta_1-\beta_1\cos\beta_1) \quad (15a)$$

$$j_s = \frac{2(\sin\beta_1-\beta_1\cos\beta_1)}{\beta_1-\sin\beta_1\cos\beta_1}\left(\frac{\sin\beta_1}{\beta_1}\right) = j_c\left(\frac{\sin\beta_1}{\beta_1}\right) \quad (16a)$$

Infinite Cylinder

$$\frac{(T-T_1)}{(T_o-T_1)} = \sum_{i=1}^{\infty}\left(\frac{2}{\beta_1}\right)\frac{J_1(\beta_1)}{J_o^2(\beta_1)+J_1^2(\beta_1)}e^{\frac{-\beta_i^2\alpha t}{r^2}}J_o\left(\beta_1\frac{x}{r}\right) \quad (17a)$$

$$\text{root equation: } N_{Bi} = \frac{\beta_1 J_1(\beta_1)}{J_o(\beta_1)} \quad (18a)$$

First term approximation:

$$\log\frac{(T-T_1)}{(T_o-T_1)} = \frac{-\beta_1^2\alpha t}{2.303r^2} + \log\left[\frac{2J_1(\beta_1)}{\beta_1(J_o^2(\beta_1)+J_1^2(\beta_1))}J_o\left(\beta_1\frac{x}{r}\right)\right] \quad (19a)$$

$$\frac{\beta_1^2\alpha}{2.303r^2} = \frac{1}{f};\quad \frac{f\alpha}{r^2} = \frac{2.303}{\beta_1^2} \quad (20a)$$

or

$$\frac{f}{r^2} = \frac{2.303}{\beta_1^2\alpha}$$

$$j = \frac{2J_1(\beta_1)}{\beta_1(J_o^2(\beta_1)+J_1^2(\beta_1))}J_o\left(\beta_1\frac{x}{r}\right) \quad (21a)$$

$$j_c = \frac{2J_1(\beta_1)}{\beta_1(J_o^2(\beta_1)+J_1^2(\beta_1))} = \frac{2}{\beta_1 J_1(\beta_1)\left(1+\frac{\beta_1^2}{N_{Bi}^2}\right)} \quad (22a)$$

$$j_m = \frac{4J_1^2(\beta_1)}{\beta_1^2(J_o^2(\beta_1)+J_1^2(\beta_1))} = j_c\frac{2J_1(\beta_1)}{\beta_1} \quad (23a)$$

$$j_s = \frac{2J_1(\beta_1)J_o(\beta_1)}{\beta_1(J_o^2(\beta_1)+J_1^2(\beta_1))} = j_c(J_o(\beta_1)) \quad (24a)$$

Table IIA. Numerical Solutions for Sphere Eq. 10a

β_1	N_{Bi}	$\frac{f\alpha}{r^2}$	j_c	j_m	j_s
0.000	0.00000	∞	1.000000	1.000000	1.000000
0.020	0.00013	5756.46273	1.000040	1.000000	0.999973
0.030	0.00030	2558.42788	1.000090	1.000000	0.999940
0.040	0.00053	1439.11568	1.000160	1.000000	0.999893
0.050	0.00083	912.03404	1.000250	1.000000	0.999833
0.060	0.00120	639.60697	1.000360	1.000000	0.999760
0.085	0.00241	318.69690	1.000723	1.000000	0.999518
0.110	0.00404	190.29629	1.001211	1.000000	0.999193
0.135	0.00608	126.34212	1.001824	0.999999	0.998784
0.160	0.00855	89.94473	1.002563	0.999999	0.998291
0.180	0.01082	71.06744	1.003245	0.999998	0.997836
0.260	0.02264	34.06191	1.006781	0.999991	0.995476
0.340	0.03883	19.91856	1.011620	0.999974	0.992242
0.420	0.05950	13.05320	1.017781	0.999940	0.988121
0.500	0.08476	9.21034	1.025283	0.999878	0.983093
0.600	0.12298	6.39607	1.036586	0.999745	0.975500
0.800	0.22303	3.59779	1.065847	0.999174	0.955740
1.000	0.35791	2.30259	1.104494	0.997917	0.929400
1.200	0.53346	1.59902	1.153260	0.995507	0.895736
1.400	0.75853	1.17479	1.212969	0.991278	0.853800
1.650	1.13096	0.84576	1.304219	0.981982	0.787958
1.950	1.77705	0.60555	1.439317	0.961328	0.685676
2.250	2.81653	0.45483	1.600330	0.923671	0.553411
2.550	4.79541	0.35411	1.775242	0.858958	0.388245
2.850	10.49531	0.28348	1.930778	0.754851	0.194757
2.910	13.33971	0.27191	1.954322	0.728485	0.154148
2.970	18.13822	0.26104	1.973801	0.700028	0.113478
3.030	28.03952	0.25080	1.988439	0.669592	0.073081
3.090	60.83910	0.24116	1.997426	0.637227	0.033336
3.110	99.40785	0.23806	1.999022	0.626026	0.020304
3.116	122.72711	0.23715	1.999356	0.622626	0.016420
3.122	160.32505	0.23624	1.999621	0.619209	0.012548
3.128	231.11014	0.23533	1.999817	0.615774	0.008690
3.134	413.75948	0.23443	1.999943	0.612321	0.004845
3.140	1972.55075	0.23354	1.999997	0.608851	0.001014
π	∞	0.23330	2.000000	0.607927	0.000000

Table IIIA. Numerical Solution for Infinite Cylinder Eq. 17a

β_1	N_{Bi}	$\frac{f\alpha}{r^2}$	j_c	j_m	j_s
0.000	0.000000	∞	1.000000	1.000000	1.000000
0.020	0.00020	5756.46273	1.000050	1.000000	0.999950
0.030	0.00045	2558.42788	1.000113	1.000000	0.999887
0.040	0.00080	1439.11568	1.000200	1.000000	0.999800
0.050	0.00125	921.03404	1.000313	1.000000	0.999687
0.060	0.00180	639.60697	1.000450	1.000000	0.999550
0.080	0.00320	359.77892	1.000800	1.000000	0.999200
0.100	0.00501	230.25851	1.001251	0.999999	0.998749
0.120	0.00721	159.90174	1.001801	0.999999	0.998198
0.140	0.00982	117.47883	1.002452	0.999998	0.997546
0.150	0.01128	102.33712	1.002815	0.999997	0.997182
0.200	0.02010	57.56463	1.005008	0.999992	0.994983
0.250	0.03150	36.84136	1.007833	0.999979	0.992147
0.300	0.04551	25.58428	1.011292	0.999957	0.988665
0.350	0.06221	18.79661	1.015390	0.999921	0.984531
0.400	0.08164	14.39116	1.020131	0.999864	0.979732
0.450	0.10390	11.37079	1.025521	0.999781	0.974258
0.600	0.18862	6.39607	1.045648	0.999294	0.953636
0.750	0.30308	4.09348	1.071851	0.998232	0.926339
0.900	0.45244	2.84270	1.104323	0.996221	0.891767
1.050	0.64445	2.08851	1.143220	0.992743	0.849178
1.200	0.89095	1.59902	1.188591	0.987103	0.797702
1.400	1.33848	1.17479	1.258814	0.974588	0.713565
1.600	2.00226	0.89945	1.338607	0.953583	0.609605
1.800	3.07874	0.71067	1.423969	0.920069	0.484130
2.000	5.15184	0.57565	1.506837	0.869031	0.337367
2.200	11.08276	0.47574	1.573174	0.795115	0.173619
2.230	13.11904	0.46303	1.580484	0.781771	0.148169
2.270	17.22961	0.44685	1.588787	0.763017	0.114099
2.310	24.78411	0.43151	1.595250	0.743155	0.080002
2.360	53.11172	0.41342	1.600416	0.716769	0.037582
2.370	68.52477	0.40994	1.601027	0.711286	0.029152
2.380	96.34846	0.40650	1.601490	0.705734	0.020745
2.390	161.69599	0.40311	1.601800	0.700115	0.012366
2.400	497.85003	0.39975	1.601956	0.694428	0.004017
2.40482	∞	0.39815	1.601975	0.691660	0.000000