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Characteristics of Transient Heat Conduction Systems in the High or Low Biot Number Region

There are five groups of parameters which must be considered in the transient heat conduction system: the characteristic dimension of the body, R, and the position variable, r; the body properties--thermal conductivity, k, and thermal capacitance, ρC_p; the external film coefficient, h; the time, t; and the dimensionless temperature change of the body, the unaccomplished temperature, $\frac{T_1 - T}{T_1 - T_o}$.

A transient heat conduction system usually consists of two basic thermal resistances, the external resistance, 1/h, and the internal resistance, $\frac{R}{k}$. The ratio of the internal to the external resistances, $\frac{hR}{k}$ is the Biot Number, N_{Bi}. Obviously the N_{Bi} is a positive number with limits between zero (no internal resistance-Newtonian system) and infinity (no external resistance). The N_{Bi} is usually needed where one of the boundary conditions of the system is a convective boundary condition (Eq 1c). For example, a very common case of aperiodic transient heating is that of a body initially at a uniform temperature, T_o, suddenly exposed to a new constant temperature, T_1. The initial and boundary conditions of such a case may be written as follows:

Initial condition:

$$T = T_o \qquad \text{at } t = 0 \qquad \text{for all } r \tag{1a}$$

Boundary condition

$$\frac{\partial T}{\partial r} = 0 \qquad \text{at } r = 0 \qquad \text{for } t > 0 \tag{1b}$$

$$-k\frac{\partial T}{\partial r} = h(T - T_1) \text{ at } r = R \qquad \text{for } t > 0 \tag{1c}$$

Assuming constant properties, only conduction heating, and no generation of internal energy, the exact solution for the three major one-dimensional heat flow shapes, the sphere, the infinite cylinder and the infinite slab having the above-mentioned initial and boundary conditions has the following general form:

$$\frac{T - T_1}{T_o - T_1} = \sum_{i=1}^{\infty} j_i \, e^{-\frac{\beta^2{}_i \alpha t}{R^2}} \tag{2}$$

The N_{Bi} appears in the transcendental equation where the positive roots, β_1, are needed for the numerical computation of the transient heat transfer system (Eq 2). For the example, the transcendental equations for the infinite slab, infinite cylinder and sphere are:

Infinite slab $\quad N_{Bi} = \beta_i \tan \beta_i \quad$ (3a)

Infinite cylinder $\quad N_{Bi} = \beta_i \frac{J_i(\beta_i)}{J_o(\beta_i)} \quad$ (3b)

Sphere $\quad N_{Bi} = 1 - \beta_i \cot \beta_i \quad$ (3c)

Theoretically, the exact numerical solution consists of an infinite number of consecutive positive roots ($\beta_1, \beta_2, \beta_3, \ldots$) all satisfying the ap-

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propriate transcendental equation. The number of root values needed to obtain a certain accuracy depends mainly on the magnitude of the time value, t. After a certain length of time has elapsed, the series-type exponential solution converges rapidly, and all the terms, except the first, (which consists of β_1) become negligible and Eq 1 simplifies to Eq 4.

$$\frac{T-T_1}{T_0-T_1} = j_1 e^{\frac{-\beta_1^2 \alpha t}{R^2}} \tag{4}$$

As mentioned previously, the N_{Bi} is the ratio of the internal resistance, $\frac{R}{k}$, to the external resistance, 1/h. The significance of a system having a high or low N_{Bi} is that one of the two thermal resistances of the system is relatively high compared to the other, and it is the higher resistance that will control the overall heat flow. When a transient heat transfer system has a high or low N_{Bi}, it is possible to derive simple useful algebraic relationships between the physical and geometric properties of the system and the temperature change.

An important and useful parameter that can be used to describe the rate of transient conduction heat transfer is the temperature-response-parameter, f. This variable was introduced by Ball (1923),[1] but the name temperature response parameter was suggested, and its dimensionless analysis was done by Kopelman (1966).[2] The temperature response-parameter is a very convenient practical term for describing the transient system since with a single term having a time dimension we can describe a 90% change in temperature difference on the linear portion of the curve when the logarithm of the temperature difference is plotted vs time. This term incorporates the thermal properties of the body, its geometric characteristic and the thermal properties of the external system, and describes the transient heat conduction system just as the overall heat transfer coefficient, U, describes the steady-state system. It has been shown (Pflug et al., 1965)[3] that the temperature-response-parameter is related to the transient heat conduction parameters in the following dimensionless form:

$$\frac{f\alpha}{R^2} = \frac{\ln(10)}{\beta_1^2} \tag{5}$$

Eq 4, which is the first term approximation of the series solution and beyond a certain time is the actual cooling or heating curve, can be described by two parameters: the temperature - response-parameter, f, the direction function, and the j parameter, which is the intercept function. The f of the straight line semilogarithmic cooling or heating curve, is independent of the point of measurement since the slope term does not contain a position variable. The j term, however, does depend on location since it contains the position variable r/R. The complete set of equations describing j and the figures where the mass-average j, the j at the geometric center and the j at the surface are plotted vs N_{Bi} are given by Pflug et al. (1965).[3]

In this paper, we shall analyze the characteristics of the temperature-response-parameter, f, in the high and low Biot Number regions. The analysis regarding the various j's in the high or low Biot Number regions is simpler and less significant, and will be mentioned briefly at the end of this analysis. These conclusions regarding the various j can be obtained immediately by observation of the figures mentioned above.

While there is a single solution for the three common geometries, the value of the root number, β_1, is different for each geometric configuration, since it is obtained for each shape by solving the appropriate transcendental equation. The root value, β_1, has very little physical meaning, and since it is a function of the N_{Bi} alone, it is more desirable to plot Eq 5 in the form of $\frac{f\alpha}{R^2}$ vs. N_{Bi} (Pflug et at., 1965).[3] In Figure 1, $\frac{f\alpha}{R^2}$ vs N_{Bi} are shown for the three basic one-dimensional heat flow geometries: infinite slab, infinite cylinder and sphere.

When the N_{Bi} is either large or small, the solution of Eq 5 described in Fig. 1 has some interesting and useful properties. Let us characterize the zones at the extreme ends of these curves.

HIGH N_{Bi}

By inspection of the transcendental equations for infinite slab, infinite cylinder and sphere (Eqs 3a, 3b, 3c), we can see that for high N_{Bi} values the root, β_1, approaches a max. value of $\pi/2$, 2.4048... and π, respectively. By substituting the values of $\pi/2$, 2.4048 and π into the exact solution (Eq 5), we find the respective asympotic values $\frac{f\alpha}{R^2}$ to be:

$\frac{4\ln(10)}{\pi^2}$	$\frac{\ln(10)}{(2.4048)^2}$	$\frac{\ln(10)}{\pi^2}$
infinite slab	infinite cylinder	sphere

By inspecting the high N_{Bi} asymptotic lines, shown in Fig. 1, we can see that for practical use the exact solution of the curve can be taken as the asymptotic value when the N_{Bi} exceeds 50. This means that in a system where N_{Bi} exceeds 50, the ratio between the f value for a sphere, infinite cylinder and infinite slab having the same characteristic dimensions will be:

$\frac{1}{\pi^2}$	:	$\frac{1}{(2.4048)^2}$	:	$\frac{1}{(\pi/2)^2}$
1	:	1.705	:	4
sphere	:	infinite cylinder	:	infinite slab

Since in high N_{Bi} system $\frac{f\alpha}{R^2}$ is constant, the following relationships for the respective geometry regarding f can be observed:

1. The f value is inversely proportional to the thermal diffusivity.

2. The f value is independent of the N_{Bi} (the root value β_1 is constant) therefore, independent of the film coefficient h. (This means that increasing h does not improve the total heat transfer in the high N_{Bi} system).

3. The f value for a defined geometry is proportional to the square of the characteristic dimension.

LOW N_{Bi}

In the low N_{Bi} system, the exterior resistance dominates, and since the internal resistance is relatively low, the temperature profile within the interior part of the system is small. Therefore, the interior temperature can be considered to be uniform. In this case, the heat balance may be written as follows:

$$V \rho C_p \frac{dT}{dt} = A h (T_s - T) \tag{6}$$

Integration of T with respect to t will give:

$$\frac{- V \rho C_p \ln(10)}{A h} \log \Big[T_1 - T\Big]_{T'}^{T''} = \Big[t\Big]_{t'}^{t''} \tag{7}$$

$$\frac{- V \rho C_p \ln(10)}{Ah} \log \frac{T_1 - T''}{T_1 - T'} = t'' - t' \tag{8}$$

By definition $t'' - t' = f$ when $\frac{T_1 - T''}{T_1 - T'} = 0.1$

Inserting this value into Eq 8 and rearranging, we obtain

$$f = \frac{\ln(10) \rho C_p}{h} \frac{V}{A} \tag{9}$$

Eq 9 and the following derivations are important in understanding and interpreting the heat flow mechanism in the low N_{Bi} system. We shall prove that Eq 9, which was derived on the single assumption that no temperature gradient exists in the interior of the body, can be derived from the exact unsteady state solution for the sphere, infinite cylinder and infinite slab. Eq 9 can be rearranged in a dimensionless form by multiplying and dividing the right side of Eq 9 by R^2/k

$$f = \frac{\ln(10) \frac{\rho C_p R^2}{k}}{\frac{h R^2}{k}} \frac{V}{A} \tag{10}$$

Since $\alpha = \frac{k}{\rho C_p}$ and $N_{Bi} = \frac{h R}{k}$, rearrangement of Eq 10 yields

$$\frac{f \alpha}{R^2} = \frac{\ln(10)}{N_{Bi} R} \frac{V}{A} \tag{11}$$

or

$$\frac{f \alpha}{R^2} = \frac{\ln(10)}{C N_{Bi}} \tag{12a}$$

$$\text{where } C - R \frac{A}{V} \tag{12b}$$

For example, the respective values of C for a sphere, infinite cylinder and infinite slab are:

$$\text{sphere} \quad \frac{R\, 4 \pi R^2}{\frac{4}{3} \pi R^3} = 3 \tag{13a}$$

$$\text{infinite cylinder} \quad \frac{R\, 2 \pi R L}{\pi R^2 L} = 2 \tag{13b}$$

$$\text{infinite slab} \quad \frac{R\, 2 a b}{2 a b R} = 1 \tag{13c}$$

by substituting the value of C in Eq 12a we get

$$\text{sphere} \quad \frac{f \alpha}{R^2} = \frac{\ln(10)}{3 N_{Bi}} \tag{14a}$$

$$\text{infinite cylinder} \quad \frac{f \alpha}{R^2} = \frac{\ln(10)}{2 N_{Bi}} \tag{14b}$$

$$\text{infinite slab} \quad \frac{f \alpha}{R^2} = \frac{\ln(10)}{N_{Bi}} \tag{14c}$$

The expressions of $\frac{f\alpha}{R^2}$ for the sphere, infinite slab as shown in eqs 14a, 14b and 14c, were obtained by the single assumption that there is no temperature gradient along the body. We can show that these equations can be obtained from the exact solution $\frac{f\alpha}{R^2} = \frac{\ln(10)}{\beta_1^2}$ (Eq 5) by expanding the appropriate transcendental equation.

FOR THE SPHERE

Transcendental equation $N_{Bi} = 1 - \beta_1 \cot \beta_1$. By expanding $\cot \beta_1$ we get:

$$N_{Bi} = 1 - \beta_1 \left(\frac{1}{\beta_1} - \frac{\beta_1}{3} - \frac{\beta_1^3}{45} \cdots \right) \tag{15}$$

Since in low N_{B1} the β_1 is small, $\beta_1^3 << \beta_1$ and Eq 15 may be approximated by:

$$N_{Bi} \approx 1 - 1 + \frac{\beta_1^2}{3} \approx \frac{\beta_1^2}{3}$$

$$\beta_1^2 \approx 3 N_{Bi} \tag{16}$$

FOR INFINITE CYLINDER

Transcendental equation, $N_{Bi} = \beta_1 \frac{J_1(\beta_1)}{J_o(\beta_1)}$, where $J_o(\beta_1)$ and $J_1(\beta_1)$ are Bessel functions of the first kind of order zero and one, respectively. For small values of β_1 the values are calculated (Hildebrand, 1963)[4] from $J_p(\beta) \approx \frac{1}{2^p p!} \beta^{\frac{p}{1}}$ and therefore when β_1 approaches zero, $J_o(\beta_1) \longrightarrow 1$ and $J_1(\beta_1) \longrightarrow \beta_1/2$, which means that for low N_{Bi} in a cylindrical shape

$$N_{Bi} \approx \beta_1 (\beta_1/2)$$

$$\beta_1^2 \approx 2 N_{Bi} \qquad (17)$$

FOR INFINITE SLAB

Transcendental equation $N_{Bi} = \beta_1 \tan \beta_1$. When β_1 is small $\beta_1 \approx \tan \beta_1$ and therefore

$$N_{Bi} \approx \beta_1^2 \qquad (18)$$

Inserting the approximate value of β_1^2 for the sphere, infinite cylinder and infinite slab, as appears in Eqs 16, 17 and 18, into the exact solution (Eq 5) yields equations identical to Eqs 14a, 14b and 14c.

The general relationship of the system properties in low N_{Bi} system is $\frac{f\,\alpha}{R^2} = \frac{\ln(10)}{C\,N_{Bi}}$. Taking the logarithm of both sides of the above expression we have

$$\log \frac{f\,\alpha}{R^2} = -\log N_{Bi} + \log \frac{\ln(10)}{C} \qquad (19)$$

This means that when $\log \frac{f\,\alpha}{R^2}$ is plotted vs. $\log N_{Bi}$, the curve in the low N_{Bi} region will approach a straight line (slope = -1). This is an asymptotic line for the curve described by the exact solution (Eq 5). By applying the appropriate numerical value of C we draw in Fig. 1 the asymptotic lines for a sphere, infinite cylinder and infinite slab. By inspecting the asymptotic lines in the low N_{Bi} region drawn in Fig. 1, we can see that for practical use, when the N_{Bi} is below 0.2 for a sphere or infinite cylinder, and below 0.1 for and infinite slab, the curve describing the exact solution (Eq 5) can be taken to be identical to the asymptotic straight line (Eq 12a), and the Eqs 14a, 14b, and 14c can be used in the heat transfer calculations.

From the analysis made for the low N_{Bi} system, we can see that:

1. From Eq 9, $f = \frac{\ln(10)\,\rho C_p}{h} \frac{V}{A}$ under the same conditions (h and N_{Bi} are constant) the ratio between the f values of various bodies, no matter how irregular their shape, is proportional to the ratio between their volume/surface area. (If we compare the f value of a sphere, infinite cylinder and infinite slab having the same characteristic dimension, we find the ratio to be equal to 1 : 1.5 : 3, respectively, which can be observed immediately through inspection of Eqs 14a, 14b and 14c.)

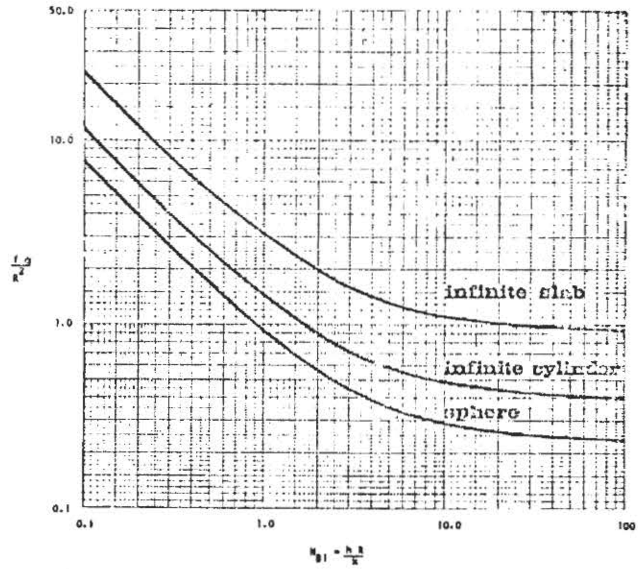

Fig. 1 N_{Bi} *vs* $\frac{f\alpha}{R^2}$ *for infinite slab, infinite cylinder and sphere*

2. For the same geometry, the f will be proportional to any chosen, but fixed, characteristic dimension to the power of 1 (rather than the power of 2 as in case of high N_{Bi} systems).

3. From Eq 9 we can see that the f is proportional to the heat capacitance, ρC_p (rather than the $\frac{\rho C_p}{k} = \frac{1}{\alpha}$ in case of high N_{Bi} systems), which shows that in low N_{Bi} systems a temperature-change comparison between two bodies should be made with respect to their thermal capacitance, ρC_p, rather than their thermal diffusivity, α. In an air-cooled room (air flow by natural convection), for example, it will take copper, regardless of its well-known high thermal conductivity and thermal diffusivity, about the same time to cool as an apple, both having the same dimension. If we compare a 3 in. diameter apple assumed to have p = 50 lb/cu ft, C_p = 0.85 Btu/lb/F and k = 0.2 Btu/hr/ft/F with a copper sphere having the same diameter and assumed to have ρ = 559 lb/cu ft, C_p = 0.0915 Btu/lb/F and k = 223 Btu/hr/ft/F both cooling under natural convection conditions (assumed h = 1.0 Btu/hr/sq ft/F, the f value for the exact solution is about 4.9 and 4.6 hr for the copper and the apple, respectively.

4. f is inversely proportional to h. (The f

was independent of the h in the case of high N_{Bi} systems, but in the low N_{Bi} system it is important to improve the exterior film coefficient in order to increase the overall rate of heat flow.)

5. $\frac{f\alpha}{R}$ vs N_{Bi} on log - log scale for any irregular shape, can be described by simply choosing an arbitrary characteristic dimension and using it consistently for the computation of $\frac{f\alpha}{R}$, N_{Bi} and C. (The lines for the different shapes will be parallel with s slope of -1, each having a different intercept with the ordinate.)

A comparision of the most important properties of low and high N_{Bi} heat conduction systems is summarized in Table 1.

DISCUSSION

The heat transfer medium in systems where food products are heated or cooled is usually steam, water or air. In all cases where water or steam is the heat transfer medium, the system can be considered to be a "high N_{Bi} system".

It is more difficult to make a similar generalization of a "low N_{Bi} system" where air is used, in spite of the fact that in many cases the system is or approaches a "low N_{Bi} system". We would like to point out, however, that the restrictions regarding $N_{Bi} < 0.2$ or $N_{Bi} > 50$ should not discourage us from evaluating the magnitude and/or the importance of the various parameters of systems where their N_{Bi} has a value between these limits. For example, we can compare the f value of two potatoes (k = 0.3 Btu/hr/ft/F) two in. and four in. dia, both cooked in an oven (h = 1 Btu/hr/sq ft/f) and boiled in water (h > 300 Btu/hr/sq ft/F). In this case, we can say that the f value of the four in. potato will be just about four times as large as the f value of the two in. potato when both are boiled in water ($N_{Bi} > 83$), and slightly more than twice as much ($N_{Bi} \approx 0.55$) if both are cooked in the oven. Similar results will be obtained if instead of the sphere we are dealing with a finite cylinder or rectangular parallelepiped.

It is quite obvious that one of the most important steps in designing and understanding a transient heat transfer system is to evaluate the N_{Bi} of the system. Knowing the N_{Bi} of the system will allow us to point out the relative importance of the various physical properties of the system, and consequently, will show the most logical steps to scale up or to improve the system. The importance of the various physical properties can be interpreted from the previous analysis summarized in Table I.

Let us take an example. A food product exposed to a high film coefficient heat transfer medium such as water or steam is being processed. The N_{Bi} in such cases is high and the f value will be almost independent of the exterior film coefficient. Practically, this means that a substantial improvement of the exterior film coefficient (by mixing or increasing the flow) will only slightly improve the rate of heating. Since f, in the high N_{Bi} systems, is proportional to R^2, it is the dimension of the body, R, which is the important criterion, and will control the rate of heat penetration. If the N_{Bi} of the system is low, the importance of the dimension, R, and the film coefficient, h, is about the same (power of 1). Mathematically speaking, we may say that by going from a high N_{Bi} to a low N_{Bi} system, the power of proportionality of the dimension was decreased from 2 to 1, while the power of proportionality of the film coefficient was increased from 0 to 1. Similar observation regarding the relative importance of the thermal properties on the rate of heat penetration can be observed from Table I.

In many cases, the ratio between the main axes of the body is not large enough (usually a ratio of 6:1 is needed) to consider the heat flow in the body to be one dimensional. But, since under similar boundary conditions the solution (Eq 2) for simple composite shapes (finite cylinder, rectangular parallelepiped, cube) are the product of the individual solution in the respective direction (Carslaw and Jaeger, 1959),[5] (Pflug, et at., 1965),[3] the previous analysis for the one-dimensional problem will hold for two or three-dimensional systems as well.

The value of j in the low or high Biot Number regions may be obtained immediately from the curves of the various j's vs N_{Bi}. Looking at these curves, we may see that j reaches a constant value for most cases, depending on the shape of the object.

CONCLUSIONS

The analysis of transient heat conduction in low and high N_{Bi} systems was conducted. It was shown that the curve, $\frac{f\alpha}{R}$ vs N_{Bi}, plotted on a log-log scale, describing the relationship between the temperature response parameter, f, and the porperties of the transient heat conduction system for each of the three basic one-dimensional geometries, is asymtotically bounded by two intersecting straight lines, the angle between which is $3/4\,\pi$. The asymtotic value of these lines are 0 and ∞, respectively. Practically, the asymptotic coincidence with the curve can be assumed to be valid when $N_{Bi} > 50$ for high N_{Bi} or $N_{Bi} < 0.2$ for the low N_{Bi}. In any system having N_{Bi} close to or beyond the points of practical coincidence, the relationships between the temperature response parameter, f, and the physical properties of the transient heat conduction system can be reduced to a simple algebraic form. It was shown that in a high N_{Bi} system, the temperature response parameter, f, is independent of N_{Bi} (or the film coefficient h) and is proportional to $\frac{C_p \rho}{k}$ (the reciprocal of the thermal diffusivity) and R^2. In a low N_{Bi} system, the f is inversely proportional to N_{Bi} (or h), but directly proportional to $C_p\rho$ (thermal capacitance) and R. The ratio between the f values of a sphere to an infinite cylinder to an infinite slab was found to be 1:1.5-1.7: 3-4, respectively (the low ratio values are for the N_{Bi} systems where the large ratio values are for the low N_{Bi} systems).

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NOMENCLATURE

A	Surface area of the body	sq ft
a	Width of slab	ft
b	Length of slab	ft
C	Coefficient, $C = R\frac{A}{V}$	dimensionless
C_p	Specific heat	Btu/lb/F
f	Temperature response parameter; the time required for a 90% change in the temperature difference on the linear portion of the heating curve	hr
h	Surface film coefficient	Btu/hr/sq ft/F
$J_n(\beta_i)$	Nth order, bessel function of the body	
j	Lag factor	dimensionless
k	Thermal conductivity of the body	Btu/hr/ft/F
L	Height of cylinder	ft
N_{Bi}	Biot Number, $\frac{h R}{k}$	dimensionless
R	Radius of sphere or cylinder; half the thickness of slab; any chosen characteristic geometric dimension	ft
r	Variable geometric position measured from the center for a given configuration	ft
T	Temperature; T, body temperature; T_o, initial temperature of the body; T_1, heating media temperature	F
t	time	hr
V	Volume	cu ft
α	Thermal diffusivity, $= \frac{k}{C_p\rho}$	$\frac{\text{sq ft}}{\text{hr}}$
β_n	Nth root of the boundary equation	
π	3.14159. . .	
ρ	Density of the body	lb_m/cu ft

Table I. Some Properties of Low and High N_{Bi} System

	High N_{Bi} $N_{Bi} > 50$			Low N_{Bi} $N_{Bi} < 0.2$		
$\frac{f\,\alpha}{R^2}$	constant			$\frac{\ln 10}{C\,N_{Bi}}$		
f	independent of N_{Bi} or h			inversely proportional to N_{Bi} or h		
f	proportional to $\frac{C_p\rho}{k}$			proportional to $C_p\rho$		
f	proportional to R^2			proportional to R		
	Sphere	Infinite cylinder	Infinite slab	Sphere	Infinite cylinder	Infinite slab
C	—	—	—	3	2	1
f ratio between the three shapes	1	1.7	4	1	1.5	3
β_1 approaches	π	2.4048	$\pi/12$	0	0	0
N_{Bi} approaches	∞	∞	∞	$\beta_1^2/3$	$\beta_1^2/2$	β_1^2