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The Relationship of the Surface, Mass Average and Geometric Center Temperatures in Transient Conduction Heat Flow

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SUMMARY

The general theoretical solution for the relationship of the surface, mass average, and the geometric center temperatures as well as the location where the temperature will represent the mass average temperature in transient heat conduction is presented. The solution is based on the first term approximation and holds for the three basic one dimensional heat flow geometries; the sphere, the infinite cylinder and the infinite slab and some of their geometrical cross products for any Biot number. The results obtained using a digital computer are presented in the form of tables and figures needed for the computation of the equation's constant. The method is illustrated by examples.

The method can be applied for other transport phenomena having the same differential equation and boundary conditions, for example, diffusion.

INTRODUCTION

In many cases knowing the mass average temperature, T_m (eq. 1), or the surface temperature, T_s, of a body exposed to transient conduction heat transfer, is important from the standpoint of heat process design and evaluation of the probable change in quality of a product.

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$$T_m = \frac{1}{m}\int_0^m T\,dm \qquad [1]$$

The mass average temperature, T_m, can be used to show the total heat removed from the object with respect to time. In the case of cooling cans of processed food, evaluation of the mass average temperature will determine hazard of undercooling where a high mass average temperature after processing may cause discoloration or softening of the product and promote growth of surviving thermophilic organisms. Overcooling the cans, the mass average temperature of the cans after cooling is too low, increases the hazard of corrosion because not enough heat remains to evaporate the water on the surface.

For cooling canned food, it is recommended that the mass average temperature be between 90° and 110°F (Charm, 1961). The evaluation of the mass average temperature is also important in estimating the cooling load for precooling systems, and the heating load in smoke houses. The surface temperature, T_s, and its integrated lethal value can be used for correlating and estimating surface phenomenon such as browning, or the hazard of freezing.

In conduction heating products, the geometric center is the most common location to measure the temperature because it represents the slowest heating (or cooling) point and because it is usually the easiest location for installing the temperature measuring device. Therefore, it is important to develop an expression where the mass average temperature or the surface temperature is a function of the temperature at the geometric center.

Solutions to heat transfer problems are basically divided into two groups—the exact type solution and the finite difference type solution. All exact type solutions require that certain physical and boundary conditions, which are usually simple, be fulfilled. Having temperature dependent thermal properties, change of phase, a nonuniform initial temperature or a variable medium temperature may cause difficulties in solving the differential equation; and for most cases there is no exact solution available.

Where postulated conditions needed for the solution of the differential equation cannot be justified, a finite difference method should be used. Exact type solutions for the mass average and surface temperatures are presented in a number of graphical forms such as the charts of Gurney-Lurie, Hottel (McAdams, 1954), Henderson *et al.* (1955), and Pflug *et al.* (1965). Charm (1961) used a finite difference method, the forward difference approximation, to solve the temperature distribution and the mass average temperature for conduction-heating canned foods during water cooling.

BASIS OF SOLUTION

In this paper we present an analytical expression which, under certain stated conditions, can be used to relate

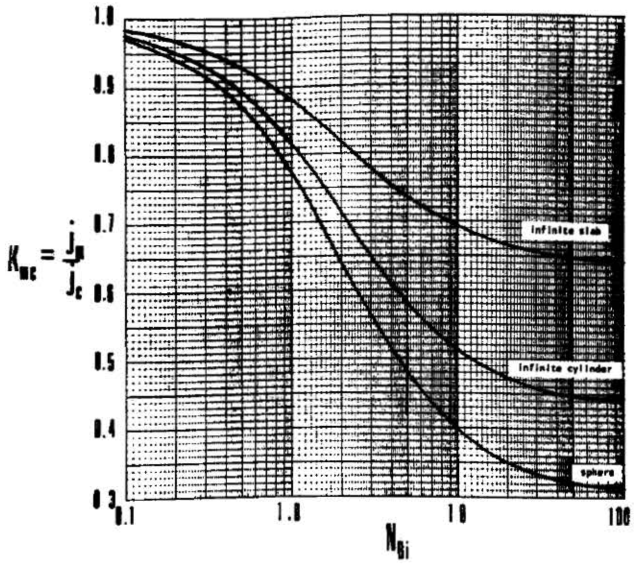

Fig. 1. $K_{mc} = j_m/j_c$ as a function of N_{Bi}.

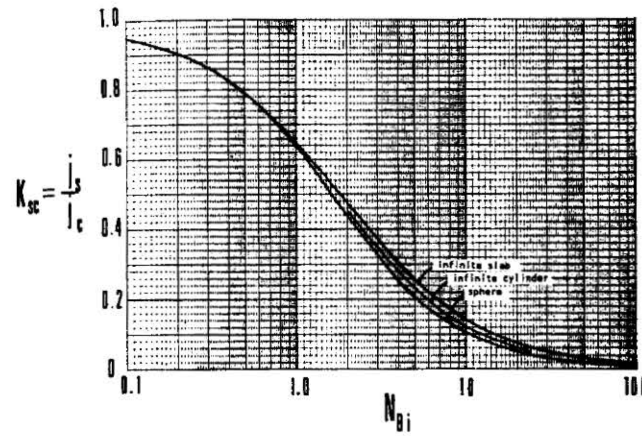

Fig. 2. $K_{sc} = j_s/j_c$ as a function of N_{Bi}.

the mass average and the surface temperature with the temperature at the geometric center. The expression, if applicable, is general and valid for any given N_{Bi} for any of the three basic one dimensional heat flow geometries—the infinite slab, the infinite cylinder, and the sphere as well as certain of their geometrical cross products. We also report the position in the body, r/R, where the temperature is the mass average temperature.

The exact solution for transient heat conduction assuming temperature independent properties, conduction heating only (no heat source or sink) for the three major one dimensional heat flow geometries, infinite slab, infinite cylinder and sphere initially at a uniform temperature and suddenly exposed to a new constant temperature, T_1, are listed in the appendix (eqs. A1, A9, and A17, respectively).

The three equations are of similar form as that shown in eq. 2.

$$\frac{T - T_1}{T_0 - T_1} = \sum_{i=1}^{\infty} j_i e^{-\frac{t}{f_i}\ln 10} \qquad [2]$$

After enough time has elapsed the series type solution (eq. 2), because of its exponential nature, converges rapidly, and all the terms except the first become negligible. Rearrangement of eq. 2 yields eq. 3.

$$\log (T - T_1) = -\frac{t}{f} + \log [j(T_0 - T_1)] \qquad [3]$$

As a rule of thumb, when the Fourier number $N_{Fo} = t\alpha/R^2$ is greater than 0.3, the aperiodic heating curve can be described by the first term approximation. This method of presenting transient conduction heating or cooling data is well known to the food technologist. Without going into detail, the two parameters of the semi-logarithmic line are the temperature response parameter, f (the direction function), and j (the intercept function). The f of the straight semi-logarithmic line is independent of the point of measurement since the slope term does not contain a position variable. The j term, however, does depend on location since it contains the position variable r/R. Both parameters depend on the N_{Bi} and the geometry of the body.

Detailed information about the first term approximation suggested by Ball (1923) can be found in numerous sources: Ball *et al.* (1957), Pflug *et al.* (1965), Pflug *et al.* (1966), and Kopelman *et al.* (1967).

THE SOLUTION AND DISCUSSION

In the following development we shall assume that sufficient time has elapsed so that we are dealing with the first term approximation. All of the heating (or cooling) curves are parallel and differ from each other only at the point of intersection with the ordinate. Only the computations for the mass average temperature are shown, because the procedure for the surface temperature is identical.

The general equation of the straight line semi-logarithmic curve (eq. 3) can be re-written for the temperature at the geometric center and the mass average temperature, respectively, as follows:

$$\log (T_1 - T_c) = -\frac{t}{f} + \log [j_c(T_1 - T_0)] \qquad [4]$$

$$\log (T_1 - T_m) = -\frac{t}{f} + \log [j_m(T_1 - T_0)] \qquad [5]$$

Subtracting eq. 4 from eq. 5 we obtain

$$\log \frac{T_1 - T_m}{T_1 - T_c} = \log \frac{j_m(T_1 - T_0)}{j_c(T_1 - T_0)} \qquad [6]$$

$$\frac{T_1 - T_m}{T_1 - T_c} = \frac{j_m}{j_c} = K_{mc} \qquad [7]$$

When we arrange eq. 7 we obtain

$$T_m = T_1 - K_{mc}(T_1 - T_c) \qquad [8]$$

The constant $K_{mc} = j_m/j_c$, is a function of the shape and the N_{Bi} and can be evaluated from the appropriate j tables or graphs.

A similar expression (eq. 9) can be obtained for the surface temperature T_s,

$$T_s = T_1 - K_{sc}(T_1 - T_c) \qquad [9]$$

$$K_{sc} = \frac{j_s}{j_c}$$

By a similar procedure we can obtain any combination of the relationship between any two of the three temperatures: T_c, T_m, T_s.

For composite bodies such as the finite cylinder, rectangular parallelepiped or cube it can be shown that their solution is the product of the respective individual solution given by eqs. A1, A9, and A17 (see appendix) hence for a solid with three dimensional heat transfer $j_{composite} = j_1\, j_2\, j_3$ (see example 1).

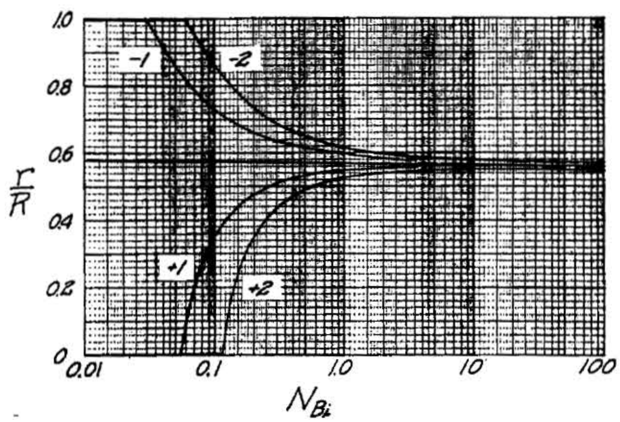

Fig. 3. r/R for infinite slab.

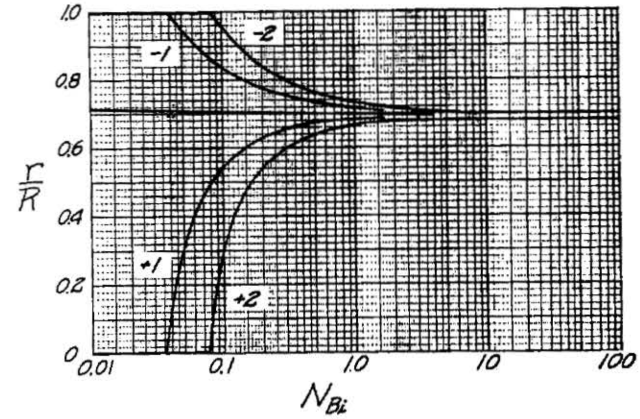

Fig. 4. r/R for infinite cylinder.

The value of j_c and j_s for each shape can be computed from its respective general j equation by substituting 0 and 1 respectively for r/R. The j_m can be obtained by integration of j with respect to the mass.

The numerical values for j_c, j_m, j_s, K_{mc} and K_{sc} for an infinite slab, an infinite cylinder and sphere vs. N_{Bi} are tabulated in tables A1, A2, and A3 (see appendix). Graphical presentation of K_{mc} and K_{sc} vs. N_{Bi} is given in Figs. 1 and 2.

The location, r/R, where the temperature is the mass average temperature can be found by equating the general j equation to j_m and solving for r/R. The resulting relationships for infinite slab, infinite cylinder and sphere are shown in eqs. 10, 11 and 12, respectively.

$$\cos\left(\beta_1 \frac{r}{R}\right) = \frac{\sin \beta_1}{\beta_1} \quad [10]$$

$$J_0\left(\beta_1 \frac{r}{R}\right) = \frac{2J_1(\beta_1)}{\beta_1} \quad [11]$$

$$\frac{\sin\left(\beta_1 \frac{r}{R}\right)}{\left(\beta_1 \frac{r}{R}\right)} = \frac{3}{\beta_1^3}(\sin \beta_1 - \beta_1 \cos \beta_1) \quad [12]$$

Only for the infinite slab (eq. 10) can r/R be solved explicitly. The solution is:

$$\frac{r}{R} = \frac{1}{\beta_1} \arccos\left(\frac{\sin \beta_1}{\beta_1}\right) \quad [13]$$

For the infinite cylinder and sphere, r/R cannot be solved explicitly because the respective equation is transcendental with respect to r/R. These equations can be solved implicitly by iteration of r/R until the two sides of the equation agree with certain prescribed values. The value of r/R as a function of N_{Bi}, for the infinite slab, infinite cylinder and sphere was com-

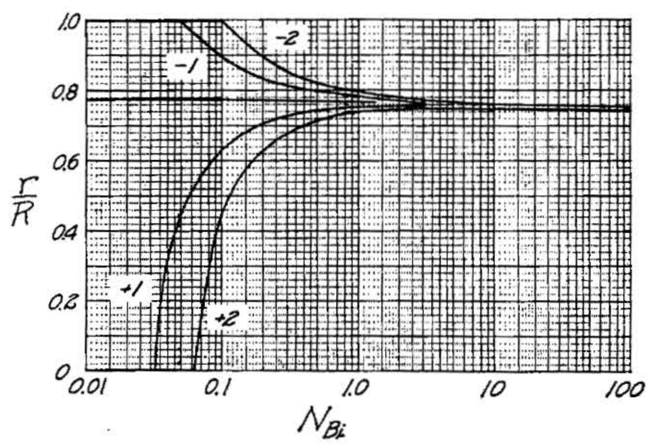

Fig. 5. r/R for sphere.

NOTATION

e	Napierian base (= 2.71826..)	
f	temperature response parameter; the time required for a 90% change in the temperature difference on the linear portion of the curve	hr
h	surface heat transfer coefficient	BTU/hr ft^2 °F
$J_n(\beta_n)$	Nth order Bessel function of first kind for the argument β_n	
j	lag factor $(T_a - T_1)/(T_o - T_1)$; j_c, lag factor at the geometric center; j_m, lag factor for the mass average; j_s, lag factor at the surface	dimensionless
K	constant; $K_{mc} = j_m/j_c$; $K_{sc} = j_s/j_c$; $K_{sm} = j_s/j_m$	dimensionless
k	thermal conductivity	BTU/hr ft °F
m	mass of product	lb_m
N_{Bi}	Biot number, hR/k	dimensionless
N_{Fo}	Fourier number $\alpha t/R^2$	dimensionless
R	radius of sphere or infinite cylinder, half the thickness of infinite slab	ft
r	variable position, distance from center of product to point of measurement	ft
T	temperature; T_o, initial temperature; T, product temperature; T_1, medium temperature; T_a, the apparent initial temperature as defined by the linear portion of the heating curve, that is the ordinate value of the asymptote of heating curve; T_c, temperature at the geometric center; T_m, mass average temperature; T_s, surface temperature	°F
t	time	hr
U	over-all heat transfer coefficient	BTU/hr ft^2 °F
α	thermal diffusivity	ft^2/hr
β_n	Nth root of the boundary equation for the particular shape.	

puted from the respective equation and is presented in tables A1, A2 and A3 (see appendix) as well as in Figs. 3, 4 and 5.

From eqs. 10, 11 and 12, it is clear that for each N_{Bi} there is a single location, r/R, where the value of j is equal to j_m. However, by allowing for a certain degree of error in j, we can show that a range of points around the exact location can represent the point of measurement of the mass average temperature. From error analysis, presented in the appendix, it can be seen that the range around the exact location for any prescribed error in j is inversely proportional to N_{Bi}. The smaller the N_{Bi} the larger the range.

This range, as a function of N_{Bi}, for a prescribed error of ±1% and ±2% in j_m ($\Delta j_m/j_m = 0.01$ and 0.02, respectively) is presented in Figs. 3, 4 and 5. For example, for the sphere, Fig. 5, where we allow the prescribed error to be 1% ($\Delta j_m/j_m = \pm 0.01$) the mass average temperature can be measured between $0.63 \leq r/R \leq 0.895$ for $N_{Bi} = .1$. This range is narrowed to $0.752 \leq r/R \leq 0.782$ for $N_{Bi} = 1$.

For composite bodies such as the finite cylinder, rectangular parallelepiped or cube, there are an infinite number of combinations of r/R in the several directions where the multiplication of the respective j's will give a fixed value. This group of points will locate the iso-j_m.

Example 1. A 3 in. diameter × 4 in. high can of food (k = 0.3 BTU/hr ft °F) is heated in a 250°F heat transfer medium. When the center reaches 240°F, what is the mass average temperature if the heat transfer medium is (a) air (h = 3 BTU/hr ft² °F), (b) water or steam.

In this specific example we will consider the can to be a two dimensional heating system where the flow of heat is both in the radial and the longitudinal directions.

$$K_{mc} = \frac{[j_{\text{infinite cylinder}}\, j_{\text{infinite slab}}]_m}{[j_{\text{infinite cylinder}}\, j_{\text{infinite slab}}]_c} \quad [14]$$

$$= (K_{mc})_{\text{infinite slab}}\, (K_{mc})_{\text{infinite cylinder}} \quad [15]$$

The N_{Bi} for the radial and longitudinal direction is 1.25 and 1.67 and the value of K_{mc} is 0.785 and 0.835, respectively. Therefore, $T_m = 250° - 0.655\ (250° - 240°) = 243.5°F$.

For heating in water or steam it is clear that the N_{Bi} is large enough that we can use the asymptote values of K_{mc} which are approximately 0.43 and 0.64, respectively, resulting in $T_m = 250° - 0.275\ (250° - 240°) = 247.3°F$.

APPENDIX

Transient heat conduction in infinite slab

$$\frac{(T - T_1)}{(T_0 - T_1)} = \sum_{i=1}^{\infty} \frac{2 \sin \beta_i}{\beta_i + \sin \beta_i \cos \beta_i} \cos\left(\beta_i \frac{r}{R}\right) e^{-\beta_i^2 \alpha t/R^2} \quad [A.1]$$

root equation:

$$N_{Bi} = \beta_i \tan \beta_i \quad [A.2]$$

1st term approximation:

$$\log \frac{(T - T_1)}{(T_0 - T_1)} = \frac{-\beta_1^2 \alpha t}{(\ln 10) R^2} + \log\left[\left(\frac{2 \sin \beta_1}{\beta_1 + \sin \beta_1 \cos \beta_1}\right) \cos\left(\beta_1 \frac{r}{R}\right)\right] \quad [A.3]$$

$$j = \frac{2 \sin \beta_1}{\beta_1 + \sin \beta_1 \cos \beta_1} \cos\left(\beta_1 \frac{r}{R}\right); \quad \frac{f\alpha}{R^2} = \frac{\ln 10}{\beta_1^2}$$

$$j_c = \frac{2 \sin \beta_1}{\beta_1 + \sin \beta_1 \cos \beta_1} \quad [A.4]$$

$$j_m = \frac{2 \sin^2 \beta_1}{\beta_1(\beta_1 + \sin \beta_1 \cos \beta_1)} = j_c\left(\frac{\sin \beta_1}{\beta_1}\right) \quad [A.5]$$

$$j_s = \frac{2 \sin^2 \beta_1}{\beta_1 + \sin \beta_1 \cos \beta_1} \cos \beta_1 = j_c \cos \beta_1 \quad [A.6]$$

Transient heat conduction in infinite cylinder

$$\frac{(T - T_1)}{(T_0 - T_1)} = \sum_{i=1}^{\infty} \left(\frac{2}{\beta_i}\right) \frac{J_1(\beta_i)}{J_0^2(\beta_i) + J_1^2(\beta_i)} J_0\left(\beta_i \frac{r}{R}\right) e^{-\beta_i^2 \alpha t/R^2} \quad [A.7]$$

root equation:

$$N_{Bi} = \beta_i \frac{J_1(\beta_i)}{J_0(\beta_i)} \quad [A.8]$$

1st term approximation:

$$\log \frac{(T - T_1)}{(T_0 - T_1)} = \frac{-\beta_1^2 \alpha t}{(\ln 10)\, R^2} + \log\left\{\frac{2 J_1(\beta_1)}{\beta_1\, [J_0^2(\beta_1) + J_1^2(\beta_1)]} J_0\left(\beta_1 \frac{r}{R}\right)\right\} \quad [A.9]$$

$$j = \frac{2 J_1(\beta_1)}{\beta_1 [J_0^2(\beta_1) + J_1^2(\beta_1)]} J_0\left(\beta_1 \frac{r}{R}\right); \quad \frac{f\alpha}{R^2} = \frac{\ln 10}{\beta_1^2}$$

$$j_c = \frac{2 J_1(\beta_1)}{\beta_1 [J_0^2(\beta_1) + J_1^2(\beta_1)]} \quad [A.10]$$

$$j_m = \frac{4 J_1^2(\beta_1)}{\beta_1^2 [J_0^2(\beta_1) + J_1^2(\beta_1)]} = j_c \frac{2 J_1(\beta_1)}{\beta_1} \quad [A.11]$$

$$j_s = \frac{2 J_1(\beta_1)}{\beta_1\, [J_0^2(\beta_1) + J_1^2(\beta_1)]} J_0(\beta_1) = j_c\, J_0(\beta_1) \quad [A.12]$$

Transient heat conduction in sphere

$$\frac{(T - T_1)}{(T_0 - T_1)} = \sum_{i=1}^{\infty} \frac{2(\sin \beta_i - \beta_i \cos \beta_i)}{\beta_i - \sin \beta_i \cos \beta_i} \frac{\sin\left(\beta_i \frac{r}{R}\right)}{\beta_i \frac{r}{R}} e^{-\beta_i^2 \alpha t/R^2} \quad [A.13]$$

root equation:

$$N_{Bi} = 1 - \beta_i \cot \beta_i \quad [A.14]$$

1st term approximation: [A.15]

$$\log \frac{(T - T_1)}{(T_0 - T_1)} = \frac{-\beta_1^2 \alpha t}{(\ln 10)\, R^2} + \log\left\{\left[\frac{2(\sin \beta_1 - \beta_1 \cos \beta_1)}{\beta_1 - \sin \beta_1 \cos \beta_1}\right]\left[\frac{\sin\left(\beta_1 \frac{r}{R}\right)}{\beta_1 \frac{r}{R}}\right]\right\}$$

$$j = \left[\frac{2(\sin \beta_1 - \beta_1 \cos \beta_1)}{\beta_1 - \sin \beta_1 \cos \beta_1}\right]\left[\frac{\sin\left(\beta_1 \frac{r}{R}\right)}{\beta_1 \frac{r}{R}}\right]; \quad \frac{f\alpha}{R^2} = \frac{\ln 10}{\beta_1^2}$$

APPENDIX (continued)

$$j_c = \frac{2(\sin \beta_1 - \beta_1 \cos \beta_1)}{\beta_1 - \sin \beta_1 \cos \beta_1} \quad [A.16]$$

$$j_m = j_c \frac{3}{\beta_1{}^3} (\sin \beta_1 - \beta_1 \cos \beta_1) \quad [A.17]$$

$$j_s = \left[\frac{2(\sin \beta_1 - \beta_1 \cos \beta_1)}{\beta_1 - \sin \beta_1 \cos \beta_1}\right] \left[\frac{\sin \beta_1}{\beta_1}\right] = j_c \frac{\sin \beta_1}{\beta_1} \quad [A.18]$$

ERROR ANALYSIS

The error analysis for only the case of the infinite slab will be shown here; similar procedures can be used for the error analysis for the infinite cylinder and the sphere.

The expression for j_m is:

$$j_m = j_c \cos \left(\beta_1 \frac{r}{R}\right) \quad [A.19]$$

differentiating j_m with respect to r/R we obtain:

$$dj_m = -j_c \sin \left(\beta_1 \frac{r}{R}\right) d \left(\frac{r}{R}\right) \quad [A.20]$$

dividing the latter equation by the former and rearranging we obtain:

$$d \left(\frac{r}{R}\right) = \frac{-1}{\beta_1 \tan \left(\beta_1 \frac{r}{R}\right)} \frac{dj_m}{j_m} \quad [A.21]$$

$$\frac{d \left(\frac{r}{R}\right)}{\frac{r}{R}} = \frac{-1}{\left(\beta_1 \frac{r}{R}\right) \tan \left(\beta_1 \frac{r}{R}\right)} \left(\frac{dj_m}{j_m}\right) \quad [A.22]$$

Table A1. Numerical solution for infinite slab.

β_1	N_{Bi}*	$\frac{f\alpha}{R^2}$*	j_c	j_m	j_s	K_{mc}	K_{sc}	$\frac{r}{R}$
0.000	0.0000 + 000	∞	1.00000	1.00000	1.00000	1.00000	1.00000	——
0.020	4.0005 − 004	5.7565 + 003	1.00007	1.00000	0.99987	0.99993	0.99980	0.57735
0.030	9.0027 − 004	2.5584 + 003	1.00015	1.00000	0.99970	0.99985	0.99955	0.57734
0.040	1.6009 − 003	1.4391 + 003	1.00027	1.00000	0.99947	0.99973	0.99920	0.57734
0.050	2.5021 − 003	9.2103 + 002	1.00042	1.00000	0.99917	0.99958	0.99875	0.57733
0.070	4.9080 − 003	4.6992 + 002	1.00082	1.00000	0.99837	0.99918	0.99755	0.57732
0.090	8.1219 − 003	2.8427 + 002	1.00135	1.00000	0.99730	0.99865	0.99595	0.57730
0.110	1.2149 − 002	1.9030 + 002	1.00202	1.00000	0.99596	0.99798	0.99396	0.57727
0.130	1.6996 − 002	1.3625 + 002	1.00282	0.99999	0.99435	0.99719	0.99156	0.57724
0.150	2.2670 − 002	1.0234 + 002	1.00375	0.99999	0.99248	0.99625	0.98877	0.57721
0.220	4.9196 − 002	4.7574 + 001	1.00806	0.99995	0.98376	0.99195	0.97590	0.57704
0.290	8.6540 − 002	2.7379 + 001	1.01399	0.99984	0.97165	0.98604	0.95824	0.57681
0.360	1.3551 − 001	1.7767 + 001	1.02154	0.99962	0.95606	0.97854	0.93590	0.57652
0.430	1.9721 − 001	1.2453 + 001	1.03068	0.99921	0.93686	0.96947	0.90897	0.57616
0.500	2.7315 − 001	9.2103 + 000	1.04140	0.99854	0.91391	0.95885	0.87758	0.57574
0.570	3.6535 − 001	7.0871 + 000	1.05364	0.99751	0.88706	0.94672	0.84190	0.57525
0.600	4.1048 − 001	6.3961 + 000	1.05935	0.99692	0.87432	0.94107	0.82534	0.57502
0.720	6.3149 − 001	4.4417 + 000	1.08476	0.99343	0.81553	0.91581	0.75181	0.57399
0.840	9.3718 − 001	3.2633 + 000	1.11388	0.98744	0.74348	0.88648	0.66746	0.57275
0.960	1.3712 + 000	2.4985 + 000	1.14586	0.97779	0.65718	0.85332	0.57352	0.57132
1.080	2.0209 + 000	1.9741 + 000	1.17933	0.96307	0.55585	0.81663	0.47133	0.56968
1.200	3.0866 + 000	1.5990 + 000	1.21223	0.94153	0.43926	0.77670	0.36236	0.56782
1.320	5.1524 + 000	1.3215 + 000	1.24162	0.91119	0.30814	0.73388	0.24818	0.56575
1.340	5.7025 + 000	1.2823 + 000	1.24591	0.90513	0.28501	0.72648	0.22875	0.56538
1.380	7.1449 + 000	1.2091 + 000	1.25380	0.89207	0.23777	0.71149	0.18964	0.56463
1.420	9.3452 + 000	1.1419 + 000	1.26062	0.87768	0.18938	0.69623	0.15023	0.56384
1.460	1.3123 + 001	1.0802 + 000	1.26616	0.86192	0.14000	0.68073	0.11057	0.56304
1.500	2.1152 + 001	1.0234 + 000	1.27024	0.84471	0.08985	0.66500	0.07074	0.56220
1.540	4.9990 + 001	9.7090 − 001	1.27265	0.82601	0.03919	0.64904	0.03079	0.56134
1.550	7.4522 + 001	9.5841 − 001	1.27297	0.82109	0.02647	0.64502	0.02079	0.56113
1.554	9.2512 + 001	9.5348 − 001	1.27306	0.81910	0.02138	0.64341	0.01680	0.56104
1.558	1.2175 + 002	9.4859 − 001	1.27314	0.81709	0.01629	0.64180	0.01280	0.56095
1.562	1.7757 + 002	9.4374 − 001	1.27319	0.81507	0.01120	0.64018	0.00880	0.56086
1.566	3.2650 + 002	9.3893 − 001	1.27322	0.81303	0.00611	0.63856	0.00480	0.56077
1.570	1.9716 + 003	9.3415 − 001	1.27324	0.81098	0.00101	0.63694	0.00080	0.56968
π/2	∞	9.3320 − 001	1.27324	0.81057	0.00000	0.63662	0.00000	0.56066

* Sign and value following it indicate exponent power to the base ten. For example, 2.5021 − 003 means 2.5021×10^{-3}.

Example 2. Apples (k = 0.2 BTU/hr ft °F), 85°F initial temperature and 2.5 in. diameter are to be placed in 35°F storage after 90% of the sensible heat is removed by the precooling system. What will be the surface temperature of the apples at the end of the precooling period using (a) 25°F air (h = 3 BTU/hr ft² °F), (b) 35°F water (h = 100 BTU/hr ft² °F).

Removing 90% of the sensible heat means that the mass average temperature, T_m, at the end of the precooling period should be 40°F. Using a procedure similar to that outlined in eqs. 3 through 8 we obtained: $T_s = T_1 - K_{sm} (T_1 - T_m)$ where $K_{sm} = j_s/j_m$. For the air and water N_{Bi} will be 1.55 and 52, respectively; using the appropriate figures and tables we find K_{sm} to be 0.51 and 0.02 resulting in surface temperatures of 32.6°F and 35.1°F.

The location of r/R where the mass average temperature can be measured with a prescribed error of $\Delta j_m/j_m = 0.02$ can be found from Fig. 5 to be $0.742 \leq r/R \leq 0.785$ and $0.745 \leq r/R \leq 0.755$ for the air and water system, respectively.

j_m/j_c and j_s/j_c reach their minimum value when $N_{Bi} \to \infty$, and approached 1 when $j_c \sim j_m \sim j_s \sim 1$ (No internal temperature gradient — Newtonian heating) for $N_{Bi} \to 0$ (Figs. 1 and 3).

Stumbo (1964) derived an equation in the same format as eq. 8. To find the constant K_{mc} he applied a laborious graphical integration using some of his previous experimental data relating iso-j lines in cylindrical cans. This result was only valid for the specific case of a body, having a cylindrical shape, exposed to a heat transfer medium which was either steam or water, with a food product in the can—a high N_{Bi} system. For this specific case, he found the average value of the constant K_{mc} to be 0.27; we found this value to be 0.275 (see example 1).

Smith *et al.* (1965) meeasured the temperature profiles of six varieties of peaches (2.3 in. average diameter) during agitated water cooling. They correlated the results using third degree orthogonal polynomials and estimated the r/R to be between 0.76 and 0.77. Using a thermal conductivity value for peaches of 0.29 BTU/hr ft °F (Smith *et al.*, 1965) and assuming an arbitrary heat transfer film coefficient for water of 200 BTU/hr ft °F, we find from Fig. 5 that r/R = 0.748.

The exact location of r/R is almost N_{Bi} independent and is practically constant for each geometry (Figs. 3, 4 and 5). The N_{Bi} established both the location, r/R, and the range of points around this exact location that can represent the mass average temperature for an allowed error. From Figs. 3, 4 and 5 it is apparent that N_{Bi} is more important in establishing the range than the exact location.

The approach presented in this paper can be adapted easily to other transport phenomena where the physical system can be described by similar differential equations and boundary conditions. For example diffusion problems, where the medium concentration is constant, can be solved by replacing the temperature with the concentration (Charm, 1963). Most diffusion problems can be considered to be high N_{Bi} systems since the exterior resistance to diffusion can usually be neglected.

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APPENDIX (concluded)

Table A2. Numerical solution for infinite cylinder.

β_1	N_{Bi}*	$\frac{f\alpha}{R^2}$*	j_c	j_m	j_s	K_{ms}	K_{sc}	$\frac{r}{R}$
0.000	0.0000 + 000	∞	1.00000	1.00000	1.00000	1.00000	1.00000	——
0.020	2.0001 − 004	5.7565 + 003	1.00005	1.00000	0.99995	0.99995	0.99990	0.70711
0.030	4.5005 − 004	2.5584 + 003	1.00011	1.00000	0.99989	0.99989	0.99978	0.70710
0.040	8.0016 − 004	1.4391 + 003	1.00020	1.00000	0.99980	0.99980	0.99960	0.70710
0.050	1.2504 − 003	9.2103 + 002	1.00031	1.00000	0.99969	0.99969	0.99938	0.70710
0.060	1.8008 − 003	6.3961 + 002	1.00045	1.00000	0.99955	0.99955	0.99910	0.70709
0.080	3.2026 − 003	3.5978 + 002	1.00080	1.00000	0.99920	0.99920	0.99840	0.70708
0.100	5.0063 − 003	2.3026 + 002	1.00125	1.00000	0.99875	0.99875	0.99750	0.70707
0.120	7.2130 − 003	1.5990 + 002	1.00180	1.00000	0.99820	0.99820	0.99640	0.70705
0.140	9.8241 − 003	1.1748 + 002	1.00245	1.00000	0.99755	0.99755	0.99511	0.70703
0.150	1.1282 − 002	1.0234 + 002	1.00282	1.00000	0.99718	0.99719	0.99438	0.70702
0.200	2.0101 − 002	5.7565 + 001	1.00501	0.99999	0.99498	0.99501	0.99002	0.70696
0.250	3.1497 − 002	3.6841 + 001	1.00783	0.99998	0.99215	0.99221	0.98444	0.70688
0.300	4.5514 − 002	2.5584 + 001	1.01129	0.99996	0.98867	0.98879	0.97763	0.70677
0.350	6.2207 − 002	1.8797 + 001	1.01539	0.99992	0.98453	0.98477	0.96961	0.70665
0.400	8.1644 − 002	1.4391 + 001	1.02013	0.99986	0.97973	0.98013	0.96040	0.70652
0.450	1.0390 − 001	1.1371 + 001	1.02552	0.99978	0.97426	0.97490	0.95001	0.70636
0.600	1.8862 − 001	6.3961 + 000	1.04565	0.99929	0.95364	0.95567	0.91200	0.70577
0.750	3.0308 − 001	4.0985 + 000	1.07185	0.99823	0.92634	0.93132	0.86424	0.70501
0.900	4.5244 − 001	2.8427 + 000	1.10432	0.99622	0.89177	0.90211	0.80752	0.70407
1.050	6.4445 − 001	2.0885 + 000	1.14322	0.99274	0.84918	0.86837	0.74280	0.70294
1.200	8.9095 − 001	1.5990 + 000	1.18859	0.98710	0.79770	0.83048	0.67113	0.70162
1.400	1.3385 + 000	1.1748 + 000	1.25881	0.97459	0.71357	0.77421	0.56686	0.69954
1.600	2.0023 + 000	8.9945 − 001	1.33861	0.95358	0.60960	0.71237	0.45540	0.69708
1.800	3.0787 + 000	7.1067 − 001	1.42397	0.92007	0.48413	0.64613	0.33999	0.69420
2.000	5.1518 + 000	5.7565 − 001	1.50684	0.86903	0.33737	0.57672	0.22389	0.69087
2.200	1.1083 + 001	4.7574 − 001	1.57317	0.79511	0.17362	0.50542	0.11036	0.68703
2.230	1.3119 + 001	4.6303 − 001	1.58048	0.78177	0.14817	0.49464	0.09375	0.68641
2.270	1.7230 + 001	4.4685 − 001	1.58879	0.76302	0.11410	0.48025	0.07181	0.68556
2.310	2.4784 + 001	4.3151 − 001	1.59525	0.74316	0.08000	0.46586	0.05015	0.68469
2.360	5.3112 + 001	4.1342 − 001	1.60042	0.71677	0.03758	0.44786	0.02348	0.68356
2.370	6.8525 + 001	4.0994 − 001	1.60103	0.71129	0.02915	0.44427	0.01821	0.68334
2.380	9.6348 + 001	4.0650 − 001	1.60149	0.70573	0.02075	0.44067	0.01295	0.68310
2.390	1.6170 + 002	4.0311 − 001	1.60180	0.70011	0.01237	0.43708	0.00772	0.68287
2.400	4.9785 + 002	3.9975 − 001	1.60196	0.69443	0.00402	0.43349	0.00251	0.68264
2.404826	∞	3.9815 − 001	1.60197	0.69166	0.00000	0.43175	0.00000	0.68253

* Sign and value following it indicate exponent power to the base ten. For example, 2.5021 − 003 means 2.5021×10^{-3}.

Table A3. Numerical solution for sphere.

β_1	N_{Bi}*	$\frac{f\alpha}{R^2}$*	j_c	j_m	j_s	K_{mc}	K_{sc}	$\frac{r}{R}$
0.000	0.0000 + 000	∞	1.00000	1.00000	1.00000	1.00000	1.00000	——
0.020	1.3334 − 004	5.7565 + 003	1.00004	1.00000	0.99997	0.99996	0.99993	0.77370
0.030	3.0002 − 004	2.5584 + 003	1.00009	1.00000	0.99994	0.99991	0.99985	0.77463
0.040	5.3339 − 004	1.4391 + 003	1.00016	1.00000	0.99989	0.99984	0.99973	0.77462
0.050	8.3347 − 004	9.2103 + 002	1.00025	1.00000	0.99983	0.99975	0.99958	0.77463
0.060	1.2003 − 003	6.3961 + 002	1.00036	1.00000	0.99976	0.99964	0.99940	0.77457
0.085	2.4095 − 003	3.1870 + 002	1.00072	1.00000	0.99952	0.99928	0.99880	0.77457
0.110	4.0366 − 003	1.9030 + 002	1.00121	1.00000	0.99919	0.99879	0.99798	0.77457
0.135	6.0824 − 003	1.2634 + 002	1.00182	1.00000	0.99878	0.99818	0.99697	0.77456
0.160	8.5479 − 003	8.9945 + 001	1.00256	1.00000	0.99829	0.99744	0.99574	0.77454
0.180	1.0823 − 002	7.1067 + 001	1.00324	1.00000	0.99784	0.99676	0.99461	0.77452
0.260	2.2636 − 002	3.4062 + 001	1.00678	0.99999	0.99548	0.99326	0.98877	0.77445
0.340	3.8834 − 002	1.9919 + 001	1.01162	0.99997	0.99224	0.98849	0.98084	0.77434
0.420	5.9503 − 002	1.3053 + 001	1.01778	0.99994	0.98812	0.98247	0.97086	0.77420
0.500	8.4756 − 002	9.2103 + 000	1.02528	0.99988	0.98309	0.97522	0.95885	0.77404
0.600	1.2298 − 001	6.3961 + 000	1.03659	0.99975	0.97550	0.96446	0.94107	0.77379
0.800	2.2303 − 001	3.5978 + 000	1.06585	0.99917	0.95574	0.93745	0.89670	0.77316
1.000	3.5791 − 001	2.3026 + 000	1.10449	0.99792	0.92940	0.90351	0.84147	0.77234
1.200	5.3346 − 001	1.5990 + 000	1.15326	0.99551	0.89574	0.86321	0.77670	0.77131
1.400	7.5853 − 001	1.1748 + 000	1.21297	0.99128	0.85380	0.81723	0.70389	0.77008
1.650	1.1310 + 000	8.4576 − 001	1.30422	0.98198	0.78796	0.75293	0.60416	0.76821
1.950	1.7771 + 000	6.0555 − 001	1.43932	0.96133	0.68568	0.66791	0.47639	0.76546
2.250	2.8165 + 000	4.5483 − 001	1.60033	0.92367	0.55341	0.57718	0.34581	0.76208
2.550	4.7954 + 000	3.5411 − 001	1.77524	0.85896	0.38824	0.48385	0.21870	0.75796
2.850	1.0495 + 001	2.8348 − 001	1.93078	0.75495	0.19476	0.39101	0.10087	0.75297
2.910	1.3340 + 001	2.7191 − 001	1.95432	0.72848	0.15415	0.37276	0.07888	0.75186
2.970	1.8138 + 001	2.6104 − 001	1.97380	0.70003	0.11348	0.35466	0.05749	0.75070
3.030	2.8040 + 001	2.5080 − 001	1.98844	0.66959	0.07308	0.33674	0.03675	0.74949
3.090	6.0839 + 001	2.4116 − 001	1.99743	0.63723	0.03334	0.31902	0.01669	0.74824
3.110	9.9408 + 001	2.3806 − 001	1.99902	0.62603	0.02030	0.31317	0.01016	0.74782
3.116	1.2273 + 002	2.3715 − 001	1.99936	0.62263	0.01642	0.31141	0.00821	0.74769
3.122	1.6033 + 002	2.3624 − 001	1.99962	0.61921	0.01255	0.30966	0.00628	0.74756
3.128	2.3111 + 002	2.3533 − 001	1.99982	0.61577	0.00869	0.30792	0.00435	0.74743
3.134	4.1376 + 002	2.3443 − 001	1.99994	0.61232	0.00485	0.30617	0.00242	0.74730
3.140	1.9726 + 003	2.3354 − 001	2.00000	0.60885	0.00101	0.30443	0.00051	0.74716
π	∞	2.3330 − 001	2.00000	0.60793	0.00000	0.30396	0.00000	0.74713

* Sign and value following it indicate exponent power to the base ten. For example, 2.5021 − 003 means 2.5021×10^{-3}.